

CLASSIFICATION OF REDUCTIVE REAL SPHERICAL PAIRS

I. THE SIMPLE CASE

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ABSTRACT. This paper gives a classification of all pairs $(\mathfrak{g}, \mathfrak{h})$ with \mathfrak{g} a simple real Lie algebra and $\mathfrak{h} < \mathfrak{g}$ a reductive subalgebra for which there exists a minimal parabolic subalgebra $\mathfrak{p} < \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ as vector sum.

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Date: September 4, 2016.

2000 Mathematics Subject Classification. 14M17, 22E15, 22F30, 53C30.

The second author was supported by ERC Advanced Investigators Grant HARG 268105.

1. INTRODUCTION

We recall that a pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ consisting of a complex reductive Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and a complex subalgebra $\mathfrak{h}_{\mathbb{C}}$ thereof is called *spherical* provided there exists a Borel subalgebra $\mathfrak{b}_{\mathbb{C}} < \mathfrak{g}_{\mathbb{C}}$ such that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} + \mathfrak{b}_{\mathbb{C}}$ as a sum of vector spaces (not necessarily direct). In particular, this is the case for symmetric pairs, that is, when $\mathfrak{h}_{\mathbb{C}}$ consists of the elements fixed by an involution of $\mathfrak{g}_{\mathbb{C}}$.

Complex spherical pairs with $\mathfrak{h}_{\mathbb{C}}$ reductive were classified by Krämer [16] for $\mathfrak{g}_{\mathbb{C}}$ simple and for $\mathfrak{g}_{\mathbb{C}}$ semi-simple by Brion [6] and Mikityuk [18].

The objective of this paper is to obtain the appropriate real version of the classification of Krämer. To be more precise, let \mathfrak{g} be a real reductive Lie algebra and $\mathfrak{h} < \mathfrak{g}$ a subalgebra. We call \mathfrak{h} *real spherical* provided there exists a minimal parabolic subalgebra $\mathfrak{p} < \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$. Being in this situation we call $(\mathfrak{g}, \mathfrak{h})$ a *real spherical pair*. The pair is said to be trivial if $\mathfrak{h} = \mathfrak{g}$.

We say that $(\mathfrak{g}, \mathfrak{h})$ is *absolutely spherical* if the complexified pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is spherical. It is easy to see (cf. Lemma 2.1) that then $(\mathfrak{g}, \mathfrak{h})$ is real spherical. In particular, real symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ are absolutely spherical and have been classified by Berger [3].

Assume that \mathfrak{g} is simple and \mathfrak{h} reductive. It is not difficult to classify also the non-symmetric absolutely spherical pairs; this is done in Table 7 at the end of the paper. The main result of this paper is then:

Theorem 1.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be a non-trivial real spherical pair for which \mathfrak{g} is simple and \mathfrak{h} reductive. Then at least one of the following statements holds:*

- (i) \mathfrak{g} is compact,
- (ii) \mathfrak{g} is complex and $(\mathfrak{g}, \mathfrak{h})$ is listed in Proposition 9.8,
- (iii) $(\mathfrak{g}, \mathfrak{h})$ is symmetric and listed by Berger (see [3], *Tableaux II*),
- (iv) $(\mathfrak{g}, \mathfrak{h})$ is absolutely spherical and isomorphic to some pair in Table 7,
- (v) $(\mathfrak{g}, \mathfrak{h})$ is isomorphic to some pair in Table 1.

Conversely, all pairs mentioned in (i)-(v) are real spherical.

Remark 1.2.

1. We use Berger's notation for the exceptional real Lie algebras. See Section 2.1.
2. There is some overlap between (iv) and (v), as it appeared more useful to include a couple of absolutely spherical cases in Table 1. This holds for case (1) which is absolutely spherical unless $p_1 + q_1 = p_2 + q_2$. Moreover, case (2) is absolutely spherical when $q = \frac{n}{2}$, case (8) is absolutely spherical when $p + q$ is odd, and case (9) is absolutely spherical if $q = \frac{n}{2}$ and $\mathfrak{f} = \mathfrak{u}(1)$.
3. The tables contain redundancies for small values of the parameters. These are mostly resolved by restricting \mathfrak{g} to

$$\begin{array}{ccccc} \mathfrak{su}(p, q) & \mathfrak{sl}(n, \mathbb{H}) & \mathfrak{sp}(p, q) & \mathfrak{so}(p, q) & \mathfrak{so}^*(2n) \\ p + q \geq 2 & n \geq 2 & p + q \geq 2 & p + q \geq 7 & n \geq 5 \end{array}$$

and $p \geq q \geq 1$.

4. In Table 1 the real spherical subalgebras which are of codimension one in an absolutely spherical subalgebra are marked with an $*$ in front of \mathfrak{h} (with the exception of (2) and (9) with $\mathfrak{f} = 0$ and $n = 2q$). See Lemma 9.1.

\mathfrak{g}	\mathfrak{h}	
(1) $\mathfrak{su}(p_1 + p_2, q_1 + q_2)$	$*\mathfrak{su}(p_1, q_1) + \mathfrak{su}(p_2, q_2)$	$(p_1, q_1) \neq (q_2, p_2)$
(2) $\mathfrak{su}(n, 1)$	$\mathfrak{su}(n - 2q, 1) + \mathfrak{sp}(q) + \mathfrak{f}$	$\mathfrak{f} \subseteq \mathfrak{u}(1)$ $1 \leq q \leq \frac{n}{2}$
(3) $\mathfrak{sl}(n, \mathbb{H})$	$\mathfrak{sl}(n - 1, \mathbb{H}) + \mathfrak{f}$	$\mathfrak{f} \subseteq \mathbb{C}$ $n \geq 3$
(4) $\mathfrak{sl}(n, \mathbb{H})$	$*\mathfrak{sl}(n, \mathbb{C})$	n odd
(5) $\mathfrak{sp}(p, q)$	$*\mathfrak{su}(p, q)$	$p \neq q$
(6) $\mathfrak{sp}(p, q)$	$*\mathfrak{sp}(p - 1, q)$	$p \geq 1$
(7) $\mathfrak{so}(2p, 2q)$	$*\mathfrak{su}(p, q)$	$p \neq q$
(8) $\mathfrak{so}(2p + 1, 2q)$	$*\mathfrak{su}(p, q)$	$p \neq q - 1, q$
(9) $\mathfrak{so}(n, 1)$	$\mathfrak{so}(n - 2q, 1) + \mathfrak{su}(q) + \mathfrak{f}$	$\mathfrak{f} \subseteq \mathfrak{u}(1)$ $2 \leq q \leq \frac{n}{2}$
(10) $\mathfrak{so}(n, 1)$	$\mathfrak{so}(n - 4q, 1) + \mathfrak{sp}(q) + \mathfrak{f}$	$\mathfrak{f} \subseteq \mathfrak{sp}(1)$ $2 \leq q \leq \frac{n}{4}$
(11) $\mathfrak{so}(n, 1)$	$\mathfrak{so}(n - 16, 1) + \mathfrak{spin}(9)$	$n \geq 16$
(12) $\mathfrak{so}(n, q)$	$\mathfrak{so}(n - 7, q) + \mathbf{G}_2$	$n \geq 7, q = 1, 2$
(13) $\mathfrak{so}(n, q)$	$\mathfrak{so}(n - 8, q) + \mathfrak{spin}(7)$	$n \geq 8, q = 1, 2, 3$
(14) $\mathfrak{so}(6, 3)$	$\mathfrak{so}(2, 0) + \mathbf{G}_2^1$	
(15) $\mathfrak{so}(7, 4)$	$\mathfrak{so}(3, 0) + \mathfrak{spin}(4, 3)$	
(16) $\mathfrak{so}^*(2n)$	$*\mathfrak{so}^*(2n - 2)$	$n \geq 5$
(17) $\mathfrak{so}^*(10)$	$*\mathfrak{spin}(6, 1)$ or $*\mathfrak{spin}(5, 2)$	
(18) \mathbf{E}_6^4	$\mathfrak{sl}(3, \mathbb{H}) + \mathfrak{f}$	$\mathfrak{f} \subseteq \mathfrak{u}(1)$
(19) \mathbf{E}_7^2	$*\mathbf{E}_6^2$ or $*\mathbf{E}_6^3$	
(20) \mathbf{F}_4^2	$\mathfrak{sp}(2, 1) + \mathfrak{f}$	$\mathfrak{f} \subseteq \mathfrak{u}(1)$

Table 1

5. For Lie algebras \mathfrak{g} of split rank one the real spherical pairs were previously described in [11], and a more explicit classification was later given in [12].

Our starting point is the following theorem which we prove in Sections 4 - 7, by making use of Dynkin's classification of the maximal subalgebras in a complex simple Lie algebra.

Theorem 1.3. *Let $(\mathfrak{g}, \mathfrak{h})$ be a real spherical pair for which \mathfrak{g} is simple and non-compact, and \mathfrak{h} is a maximal reductive subalgebra. Then $(\mathfrak{g}, \mathfrak{h})$ is absolutely spherical.*

Using Krämer's list [16] we then also obtain the following lemma.

Lemma 1.4. *Let \mathfrak{g} be a non-compact simple real Lie algebra without complex structure and $\mathfrak{h} \subsetneq \mathfrak{g}$ be a maximal reductive subalgebra which is spherical. Then either \mathfrak{h} is a symmetric subalgebra of \mathfrak{g} or a real form of $\mathfrak{sl}(3, \mathbb{C}) \subseteq \mathbf{G}_2^{\mathbb{C}}$ or $\mathbf{G}_2^{\mathbb{C}} \subseteq \mathfrak{so}(7, \mathbb{C})$.*

In order to complete the classification we use the following criterion, see Proposition 2.9 and Corollary 2.10: If $(\mathfrak{g}, \mathfrak{h})$ is real spherical with \mathfrak{h} reductive there exists a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u} \supset \mathfrak{p}$ such that for every reductive subalgebra $\mathfrak{h}' \subset \mathfrak{h}$, which is also spherical in \mathfrak{g} , one has

$$(1.1) \quad \mathfrak{h} = \mathfrak{h}' + \mathfrak{l} \cap \mathfrak{h}.$$

In other words (1.1) provides a factorization in the sense of Onishchik. It is not too hard to determine all $\mathfrak{l} \cap \mathfrak{h}$ for maximal \mathfrak{h} (see Tables 4 and 5). This allows us to conclude the classification by means of Onishchik's list of factorizations of complex simple Lie algebras (see Proposition 2.5).

This paper will serve as the starting point for a follow up second part which classifies all real spherical reductive subalgebras of semi-simple Lie algebras.

2. GENERALITIES

In the sequel \mathfrak{g} will always refer to a real reductive Lie algebra and $\mathfrak{h} < \mathfrak{g}$ shall be an algebraic subalgebra. The Lie algebra \mathfrak{h} is called *real spherical* provided there exists a minimal parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p}.$$

The pair $(\mathfrak{g}, \mathfrak{h})$ is then referred to as a *real spherical pair*.

This gives us the *dimension bound* for a real spherical subalgebra $\mathfrak{h} < \mathfrak{g}$:

$$(2.1) \quad \dim \mathfrak{h} \geq \dim \mathfrak{n} = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{a} - \dim \mathfrak{m}).$$

Further we record the obvious but nevertheless sometimes useful *rank inequality*

$$(2.2) \quad \text{rank}_{\mathbb{R}} \mathfrak{g} \geq \text{rank}_{\mathbb{R}} \mathfrak{h}.$$

A pair $(\mathfrak{g}, \mathfrak{h})$ of a complex Lie algebra and a complex subalgebra is called *complex spherical* or just *spherical* if it is real spherical when regarded as a pair of real Lie algebras. Note that in this case the minimal parabolic subalgebras of \mathfrak{g} are precisely the Borel subalgebras.

Given a pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ of a complex Lie algebra and a subalgebra, a *real form* of it is a pair $(\mathfrak{g}, \mathfrak{h})$ of a real Lie algebra and a subalgebra such that \mathfrak{g} and \mathfrak{h} are real forms of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$, respectively.

We recall from [15] Lemma 2.1:

Lemma 2.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be a real form of a complex spherical pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then $(\mathfrak{g}, \mathfrak{h})$ is real spherical (and hence absolutely spherical).*

Lemma 2.2. *Let $(\mathfrak{g}, \mathfrak{h})$ be a real form of a complex symmetric pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ with \mathfrak{g} semi simple. Let σ be the involution of $\mathfrak{g}_{\mathbb{C}}$ with fix point algebra $\mathfrak{h}_{\mathbb{C}}$. Then σ preserves \mathfrak{g} . In particular, $(\mathfrak{g}, \mathfrak{h})$ is symmetric.*

Proof. Let $\mathfrak{q} \subset \mathfrak{g}$ be the orthogonal complement of \mathfrak{h} with respect to the Cartan-Killing form of \mathfrak{g} . Then $\mathfrak{q}_{\mathbb{C}}$ is the orthogonal complement of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$ with respect to the Cartan-Killing form of $\mathfrak{g}_{\mathbb{C}}$. On the other hand $\mathfrak{q}_{\mathbb{C}}$ is the -1 -eigenspace of σ . The assertion follows. \square

Fix \mathfrak{g} and let $G_{\mathbb{C}}$ be a linear complex algebraic group with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We denote by G the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra \mathfrak{g} . For an algebraic subalgebra $\mathfrak{h} < \mathfrak{g}$ we denote by $H < G$ the corresponding analytic subgroup.

Let $P = MAN < G$ be a minimal parabolic subgroup. Then $Z := G/H$ is called real spherical provided that $(\mathfrak{g}, \mathfrak{h})$ is real spherical, which means that there is an open P -orbit on Z .

2.1. Notation for classical and exceptional groups. If $\mathfrak{g}_{\mathbb{C}}$ is classical, then $G_{\mathbb{C}}$ shall be the corresponding classical group, i.e. $G_{\mathbb{C}} = \text{SL}(n, \mathbb{C}), \text{SO}(n, \mathbb{C}), \text{Sp}(n, \mathbb{C})$. To avoid confusion let us stress that we use the notation $\text{Sp}(n, \mathbb{R}), \text{Sp}(n, \mathbb{C})$ to indicate that the underlying classical vector space is $\mathbb{R}^{2n}, \mathbb{C}^{2n}$. Further $\text{Sp}(n)$ denotes the compact real form of $\text{Sp}(n, \mathbb{C})$ and likewise the underlying vector space for $\text{Sp}(p, q)$ is \mathbb{C}^{2p+2q} .

By $\mathrm{SL}(n, \mathbb{H}) \subset \mathrm{SL}(2n, \mathbb{C})$ and $\mathrm{SO}^*(2n) \subset \mathrm{SO}(2n, \mathbb{C})$ we denote the subgroups of elements g which satisfy

$$gJ = J\bar{g}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Another standard notation for $\mathrm{SL}(n, \mathbb{H})$ is $\mathrm{SU}^*(2n)$.

We denote by $\mathrm{O}(p, q)$ the indefinite orthogonal group on \mathbb{R}^{p+q} . The identity component of $\mathrm{O}(p, q)$ is denoted by $\mathrm{SO}(p, q)$.

For exceptional Lie algebras we use the notation of Berger, [3] p. 117, and write $\mathbf{E}_6^{\mathbb{C}}, \mathbf{E}_7^{\mathbb{C}}$ etc. for the complex simple Lie algebras of type E_6, E_7 etc., and $\mathbf{E}_6, \mathbf{E}_7$ etc. for the corresponding compact real forms. For the non-compact real forms we write

$$\begin{array}{ll} \mathbf{E}_6^1, \mathbf{E}_6^2, \mathbf{E}_6^3, \mathbf{E}_6^4 & \text{for} \quad \text{E I, E II, E III, E IV} \\ \mathbf{E}_7^1, \mathbf{E}_7^2, \mathbf{E}_7^3 & \text{for} \quad \text{E V, E VI, E VII} \\ \mathbf{E}_8^1, \mathbf{E}_8^2 & \text{for} \quad \text{E VIII, E IX} \\ \mathbf{F}_4^1, \mathbf{F}_4^2 & \text{for} \quad \text{F I, F II} \end{array}$$

and finally \mathbf{G}_2^1 for \mathbf{G} , the unique non-compact real form of $\mathbf{G}_2^{\mathbb{C}}$.

By slight abuse of notation we also denote the simply connected Lie groups with these Lie algebras by the same symbols.

2.2. Factorizations of reductive groups. Let \mathfrak{h} be a reductive Lie algebra. Then a triple $(\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2)$ is called a *factorization of \mathfrak{h}* if \mathfrak{h}_1 and \mathfrak{h}_2 are reductive subalgebras of \mathfrak{h} and

$$(2.3) \quad \mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2.$$

It is called trivial if one of the factors equals \mathfrak{h} .

Likewise if H is a connected reductive group and H_1 and H_2 are connected subgroups of H , then we call (H, H_1, H_2) a *factorization of H* provided that

$$(2.4) \quad H = H_1 H_2.$$

Proposition 2.3. (Onishchik) *Let H be a connected reductive group and H_1, H_2 reductive subgroups of H . Then the following are equivalent:*

- (1) $(\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2)$ is a factorization of \mathfrak{h} .
- (2) (H, H_1, H_2) is a factorization of H .
- (3) $H_1 x H_2 \subset H$ is open for some $x \in H$.

Proof. We refer to [1], Prop. 4.4, for the equivalence of (1) and (2). It is obvious that (2.4) implies $H_1 x H_2 = H$ for all x , and hence in particular (2) implies (3).

Assume (3), then (1) is valid for the pair of \mathfrak{h}_1 and $\mathrm{Ad}(x)\mathfrak{h}_2$. Hence (2) holds for the pair of H_1 and xH_2x^{-1} . This implies $H = H_1 x H_2$ and thus $H_1 \times H_2$ acts transitively on H , that is, (2) holds for H_1, H_2 . \square

As a consequence we obtain the following result. Here we call a subalgebra of \mathfrak{g} compact if it generates a compact subgroup in the adjoint group of \mathfrak{g} .

Lemma 2.4. *Every factorization of a non-compact semisimple Lie algebra \mathfrak{g} by a reductive and a compact subalgebra is trivial.*

Proof. Let $\mathfrak{g} = \mathfrak{h}_1 + \mathfrak{h}_2$ be as assumed. Without loss of generality we may assume that $\mathfrak{h}_2 = \mathfrak{k}$ is a maximal compact subalgebra. Proposition 2.3 implies that G/K is homogeneous for H_1 , i.e. $H_1/H_1 \cap K = G/K$. After replacing K by a conjugate we may assume that $H_1 \cap K$ is maximal compact in H_1 . But then $H_1 = G$ and the factorization is trivial. \square

Factorizations of simple complex Lie algebras were classified in [19] as follows.

Proposition 2.5. (Onishchik) *Let \mathfrak{g} be a complex simple Lie algebra and let $\mathfrak{g} = \mathfrak{h}_1 + \mathfrak{h}_2$ where \mathfrak{h}_1 and \mathfrak{h}_2 are proper reductive complex subalgebras of \mathfrak{g} . Then, up to interchanging \mathfrak{h}_1 and \mathfrak{h}_2 , the triple $(\mathfrak{g}, \mathfrak{h}_1, \mathfrak{h}_2)$ is isomorphic to a triple in Table 2.*

\mathfrak{g}	\mathfrak{h}_1	\mathfrak{h}_2	$\mathfrak{h}_1 \cap \mathfrak{h}_2$
$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{sl}(2n-1, \mathbb{C}) + \mathfrak{f}, \mathfrak{f} \subseteq \mathbb{C}$	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n-1, \mathbb{C}) + \mathfrak{f} \quad n \geq 2$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n-1, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) + \mathfrak{f}, \mathfrak{f} \subseteq \mathbb{C}$	$\mathfrak{sl}(n-1, \mathbb{C}) + \mathfrak{f} \quad n \geq 4$
$\mathfrak{so}(4n, \mathbb{C})$	$\mathfrak{so}(4n-1, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{C}) + \mathfrak{f}, \mathfrak{f} \subseteq \mathfrak{sp}(1, \mathbb{C})$	$\mathfrak{sp}(n-1, \mathbb{C}) + \mathfrak{f} \quad n \geq 2$
$\mathfrak{so}(7, \mathbb{C})$	$\mathfrak{so}(5, \mathbb{C}) + \mathfrak{f}, \mathfrak{f} \subseteq \mathbb{C}$	$G_2^{\mathbb{C}}$	$\mathfrak{sl}(2, \mathbb{C}) + \mathfrak{f}$
$\mathfrak{so}(7, \mathbb{C})$	$\mathfrak{so}(6, \mathbb{C})$	$G_2^{\mathbb{C}}$	$\mathfrak{sl}(3, \mathbb{C})$
$\mathfrak{so}(8, \mathbb{C})$	$\mathfrak{so}(5, \mathbb{C}) + \mathfrak{f}, \mathfrak{f} \subseteq \mathfrak{sp}(1, \mathbb{C})$	$\mathfrak{spin}(7, \mathbb{C})$	$\mathfrak{sl}(2, \mathbb{C}) + \mathfrak{f}$
$\mathfrak{so}(8, \mathbb{C})$	$\mathfrak{so}(6, \mathbb{C}) + \mathfrak{f}, \mathfrak{f} \subseteq \mathbb{C}$	$\mathfrak{spin}(7, \mathbb{C})$	$\mathfrak{sl}(3, \mathbb{C}) + \mathfrak{f}$
$\mathfrak{so}(8, \mathbb{C})$	$\mathfrak{so}(7, \mathbb{C})$	$\mathfrak{spin}(7, \mathbb{C})$	$G_2^{\mathbb{C}}$
$\mathfrak{so}(8, \mathbb{C})$	$\mathfrak{spin}(7, \mathbb{C})_+$	$\mathfrak{spin}(7, \mathbb{C})_-$	$G_2^{\mathbb{C}}$
$\mathfrak{so}(16, \mathbb{C})$	$\mathfrak{so}(15, \mathbb{C})$	$\mathfrak{spin}(9, \mathbb{C})$	$\mathfrak{spin}(7, \mathbb{C})$

Table 2

Remark 2.6.

- (1) The spin representation embeds $\mathfrak{spin}(7, \mathbb{C})$ into $\mathfrak{so}(8, \mathbb{C})$ and there are two conjugacy classes of this subgroup. The subscripts in line 9 of Table 2 indicate that the factorization involves both conjugacy classes.
- (2) In all cases \mathfrak{h}_1 is given up to conjugation in \mathfrak{g} . Once \mathfrak{h}_1 is fixed, there is only one $\text{Ad}(H_1)$ -conjugacy class of \mathfrak{h}_2 in \mathfrak{g} for which the factorization is valid, except in lines 6–8, where there are exactly two such conjugacy classes, provided by $\mathfrak{spin}_{\pm}(7, \mathbb{C})$.
- (3) Observe that symplectic or exceptional Lie algebras do not admit factorizations.

2.3. Towers of spherical subgroups. Let $Z = G/H$ be a real spherical space and $P < G$ a minimal parabolic subgroup such that PH is open in G . Then the local structure theorem of [14] asserts that there is a parabolic subgroup $Q \supset P$ with Levi decomposition $Q = L \ltimes U$ such that:

- (1) $PH = QH$.
- (2) $Q \cap H = L \cap H$.
- (3) $L_n < L \cap H$ where $L_n \triangleleft L$ is the normal subgroup with Lie algebra \mathfrak{l}_n , the sum of all non-compact ideals of \mathfrak{l} .

We refer to Q as a Z -adapted parabolic.

Remark 2.7. In the special cases where Z is complex spherical note that $L_n = [L, L]$.

Lemma 2.8. *Let $H \subset G$ be reductive and real spherical, and let $Q = LU$ be adapted to G/H and P , then $P \cap H$ is a minimal parabolic subgroup in $L \cap H$.*

Proof. Since $P \cap L$ is a minimal parabolic subgroup in L , it follows from (3) above that $P \cap L \cap H$ is a minimal parabolic subgroup in $L \cap H$. Since $P \subset Q$ it follows from (2) above that $P \cap H = P \cap L \cap H$. \square

Proposition 2.9. *Let $H' \subset H \subset G$ be subgroups such that H is reductive and G/H is real spherical, and let $Q = LU$ be adapted to G/H . Then G/H' is real spherical if and only if H/H' is real spherical for the action of $L \cap H$.*

Proof. Assume G/H' is real spherical. Then for some $x \in G$ the set PxH' is open in G and intersects non-trivially with the open set PH . It follows that PyH' is open in G for some $y \in H$. Then the intersection $(PyH') \cap H = (P \cap H)yH'$ is open in H . By Lemma 2.8 this implies H/H' is real spherical for $L \cap H$.

Conversely, it is clear that if $(PyH') \cap H$ is open in H for some $y \in H$, then PyH' is open in G . \square

Corollary 2.10. *Let $H' \subset H \subset G$ be reductive subgroups and let Q be as above. If $Z' = G/H'$ is real spherical then $(H, H', L \cap H)$ is a factorization of H , that is,*

$$(2.5) \quad H = H'(L \cap H).$$

Conversely, if $Q = P$ then (2.5) implies that Z' is real spherical.

Proof. It follows from Proposition 2.9 that $(L \cap H)xH'$ is open in H for some $x \in H$. Then (2.5) follows from Proposition 2.3. Conversely, (2.5) implies that $H'Q = HQ$, and hence Z' is spherical if $Q = P$. \square

We recall from [13], Prop. 9.1, the following consistency relation of adapted parabolics.

Lemma 2.11. *Let $Z = G/H$ be a real form of a complex spherical space $Z_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$. Let $P < G$ be a minimal parabolic subgroup and $B_{\mathbb{C}} < G_{\mathbb{C}}$ a Borel subgroup such that $B_{\mathbb{C}} \subset P_{\mathbb{C}}$ and $B_{\mathbb{C}}H_{\mathbb{C}} \subset G_{\mathbb{C}}$ open. Let $\mathcal{Q}_{\mathbb{C}} \supset B_{\mathbb{C}}$ be the $Z_{\mathbb{C}}$ -adapted parabolic subgroup of $G_{\mathbb{C}}$ and $Q \supset P$ the Z -adapted parabolic subgroup of G . Then*

$$Q_{\mathbb{C}} = \mathcal{Q}_{\mathbb{C}}M_{\mathbb{C}}.$$

Remark 2.12. In case Z is absolutely spherical one can read off the adapted parabolic $\mathcal{Q}_{\mathbb{C}}$ from the Luna diagram of $Z_{\mathbb{C}}$ (see [5]). The Luna diagram consists of the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ with additional information. In particular the roots corresponding to the adapted Levi $\mathcal{L}_{\mathbb{C}} < \mathcal{Q}_{\mathbb{C}}$ are the uncircled elements in the Luna diagram where "uncircled" means no circle around, above, or below a knot in the underlying Dynkin diagram. Combining this information with the Satake diagram of \mathfrak{g} then gives us the structure of L via Lemma 2.11.

In view Remark 2.7 we have

$$(2.6) \quad [\text{Lie}(\mathcal{L}_{\mathbb{C}}), \text{Lie}(\mathcal{L}_{\mathbb{C}})] \subset \mathfrak{h}_{\mathbb{C}}$$

and in particular

$$(2.7) \quad [\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{h} \quad \text{in case } Q_{\mathbb{C}} = \mathcal{Q}_{\mathbb{C}}.$$

2.4. The case where \mathfrak{g} is a quasi-split real form of $\mathfrak{g}_{\mathbb{C}}$. Recall that \mathfrak{g} is called quasi-split if the complexification $\mathfrak{p}_{\mathbb{C}}$ of \mathfrak{p} is a Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$. An equivalent way of saying this is that \mathfrak{m} is abelian. The following is clear.

Lemma 2.13. *Let $(\mathfrak{g}, \mathfrak{h})$ be a real form of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and assume that \mathfrak{g} is quasi-split. Then $(\mathfrak{g}, \mathfrak{h})$ is real spherical if and only if $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is spherical.*

3. THE DYNKIN SCHEME OF MAXIMAL REDUCTIVE SUBGROUPS OF CLASSICAL GROUPS

Let $G_{\mathbb{C}}$ be a complex classical group and let V be the standard representation space attached to $G_{\mathbb{C}}$, i.e. $V = \mathbb{C}^n$ for $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ or $\mathrm{SO}(n, \mathbb{C})$, and $V = \mathbb{C}^{2n}$ for $G_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C})$. According to Dynkin [9], there are three possible types of a connected maximal complex reductive subgroup $H_{\mathbb{C}}$ of $G_{\mathbb{C}}$.

3.1. Type I: The action of $H_{\mathbb{C}}$ on V is reducible. Up to conjugation $H_{\mathbb{C}}$ is one of the following subgroups, which are all symmetric:

3.1.1. $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$. Here $H_{\mathbb{C}} = S(\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}))$, $n = n_1 + n_2$, $n_i > 0$.

3.1.2. $G_{\mathbb{C}} = \mathrm{SO}(n, \mathbb{C})$. Either $H_{\mathbb{C}} = \mathrm{SO}(n_1, \mathbb{C}) \times \mathrm{SO}(n_2, \mathbb{C})$ with $n = n_1 + n_2$, $n_i > 0$ or n is even and $H_{\mathbb{C}} = \mathrm{GL}(n/2, \mathbb{C})$. In the first case, the defining bilinear form on $G_{\mathbb{C}}$ restricts non-trivially to the factors $\mathbb{C}^n = V = V_1 + V_2 = \mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}$. In the second case $V = V_1 \oplus V_1^*$ for V_1 the standard representation of $\mathrm{GL}(n/2, \mathbb{C})$ and both factors V_1 and V_1^* are isotropic.

3.1.3. $G_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C})$. Here $H_{\mathbb{C}} = \mathrm{Sp}(n_1, \mathbb{C}) \times \mathrm{Sp}(n_2, \mathbb{C})$ with $n = n_1 + n_2$, $n_i > 0$, or $H_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$. In the first case, the defining bilinear form on $G_{\mathbb{C}}$ restricts non-trivially to the factors $\mathbb{C}^{2n} = V = V_1 + V_2 = \mathbb{C}^{2n_1} \oplus \mathbb{C}^{2n_2}$. In the second case $V = V_1 \oplus V_1^*$ for V_1 the standard representation of $\mathrm{GL}(n, \mathbb{C})$ and both factors V_1 and V_1^* are Lagrangian.

3.2. Type II: The action of $H_{\mathbb{C}}$ on V is irreducible, but $\mathfrak{h}_{\mathbb{C}}$ is not simple.

3.2.1. $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$. Here $H_{\mathbb{C}} = \mathrm{SL}(r, \mathbb{C}) \otimes \mathrm{SL}(s, \mathbb{C})$ and $\mathbb{C}^n = \mathbb{C}^r \otimes \mathbb{C}^s$ with $rs = n$ and $2 \leq r \leq s$.

3.2.2. $G_{\mathbb{C}} = \mathrm{SO}(n, \mathbb{C})$. Here there are two possibilities. The first is $H_{\mathbb{C}} = \mathrm{SO}(r, \mathbb{C}) \otimes \mathrm{SO}(s, \mathbb{C})$ acting on $\mathbb{C}^n = \mathbb{C}^r \otimes \mathbb{C}^s$ with $n = rs$, $3 \leq r \leq s$, and $r, s \neq 4$. The second case is $H_{\mathbb{C}} = \mathrm{Sp}(r, \mathbb{C}) \otimes \mathrm{Sp}(s, \mathbb{C})$ acting on $\mathbb{C}^n = \mathbb{C}^{2r} \otimes \mathbb{C}^{2s}$ with $n = 4rs$ and $1 \leq r \leq s$.

3.2.3. $G_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C})$. Here $H_{\mathbb{C}} = \mathrm{Sp}(r, \mathbb{C}) \otimes \mathrm{SO}(s, \mathbb{C})$ and $\mathbb{C}^{2n} = \mathbb{C}^{2r} \otimes \mathbb{C}^s$ with $n = rs$ and $r \geq 1$, $s \geq 3$. Moreover it is requested that $s \neq 4$ unless if $r = 1$.

3.3. Type III: The action of $H_{\mathbb{C}}$ on V is irreducible and $\mathfrak{h}_{\mathbb{C}}$ is simple. For this type the different cases are listed in [9], Thm. 1.5. However, we do not need this list.

3.4. Dynkin types in G .

Definition 3.1. Let G be a real classical group and $H < G$ a connected reductive subgroup. We say that H is of type I, II, or III, provided $H_{\mathbb{C}}$ is of that type in $G_{\mathbb{C}}$.

Remark 3.2. Suppose that V is the complexification of a real vector space $V_{\mathbb{R}}$ and $G < \mathrm{GL}(V_{\mathbb{R}})$. Suppose in addition that H is a complex subgroup of G , that is, there exists a complex structure $J_{\mathbb{R}}$ on $V_{\mathbb{R}}$ such that H is a complex subgroup of $\mathrm{GL}(V_{\mathbb{R}}, J_{\mathbb{R}})$. Then H is of type I. Indeed $H_{\mathbb{C}} \simeq H \times \overline{H}$ and the action of $H_{\mathbb{C}}$ on $V_{\mathbb{C}} \simeq (V_{\mathbb{R}}, J_{\mathbb{R}}) \oplus (V_{\mathbb{R}}, -J_{\mathbb{R}})$ is reducible.

3.5. Bilinear forms on prehomogeneous vector spaces. Let G be an algebraic group over \mathbb{C} and ρ be a finite-dimensional representation of G on a complex vector space V . The triplet (G, ρ, V) is called a *prehomogeneous vector space*, if $G \times \mathrm{GL}(1, \mathbb{C})$ has a Zariski open orbit in V .

Let now V be an irreducible representation of a reductive group G with center at most one-dimensional, and let G' denote the semisimple part of G . A necessary condition for (G, ρ, V) to be prehomogeneous is that G' satisfies

$$(3.1) \quad \dim(G') \geq \dim(V) - 1.$$

Two triplets (G_1, ρ_1, V_1) and (G_2, ρ_2, V_2) are said to be *equivalent* if there is a linear isomorphism $\psi : V_1 \rightarrow V_2$ such that $\widehat{\psi}(\rho_1(G_1)) = \rho_2(G_2)$ under the induced map $\widehat{\psi} : \mathrm{GL}(V_1) \rightarrow \mathrm{GL}(V_2)$.

With respect to this notion, (G, ρ, V) and $(H, \rho \circ \tau, V)$ are equivalent whenever $\tau : H \rightarrow G$ is a surjective homomorphism. In particular, (G, ρ, V) is always equivalent to (G, ρ^*, V^*) where ρ^* is dual to ρ .

Proposition 3.3. *Let (ρ, V) be an irreducible representation of a simple group G . The triplet (G, ρ, V) satisfies (3.1) if and only if it is equivalent to a triplet listed in Table 3. Those triplets which give rise to prehomogeneous vector spaces are marked in the column ‘preh’.*

Proof. The prehomogeneous vector spaces in Table 3 were determined in [2]. The additional triplets follow from Theorem 54 in [20]. \square

Table 3 divides into two parts: Each triplet listed in the first part represents a series of vector spaces while a triplet in the second part is only defined for a certain dimension. We call a triplet (G, ρ, V) *classical* or *sporadic* depending on whether it is equivalent to a triplet of the former or the latter type.

Lemma 3.4. *Let (G, ρ, V) belong to Table 3. If there exists a non-degenerate symmetric (resp. skew symmetric) G -invariant bilinear form on V , then the triplet is marked by 1 (resp. 2) in the last column of the table. Otherwise, if there is no invariant form, it is marked by 0.*

Proof. There exists a G -invariant bilinear form β on V if and only if $V \simeq V^*$ as a G representation. In this case, $V \otimes V$ must contain the trivial representation. Further, β is symmetric (skew) if and only if the trivial representation is contained in the subspace of symmetric (alternating) 2-tensors.

Let w_0 be the longest element of the Weyl group of G and assume that V has highest weight λ . Since V^* has highest weight $-w_0\lambda$, V is self-dual if and only if $-w_0\lambda = \lambda$. Hence, by inspecting the effect of $-w_0$ on the Dynkin diagram, we deduce whether the mark for V is zero or nonzero.

Now assume that V has mark $\neq 0$, i.e. it carries a G -invariant form β . Note that in case that (G, V) is a prehomogeneous vector space, the ring $\mathbb{C}[V]^{G \times \mathrm{GL}(1, \mathbb{C})} \simeq \mathrm{Sym}(V)^{G \times \mathrm{GL}(1, \mathbb{C})}$ is either zero or generated by one element (the *relative invariant*). Hence β is symmetric if and only if the relative invariant of V has degree 2. It can be determined from [20, §7] whether or not this is the case.

We are left with determining the symmetry type of β for the triplets that are not prehomogeneous: In case 1, the Cartan-Killing form is a symmetric invariant form. In case 6, let

	G	ρ	$\dim(V)$	preh.	inv. form
1.	G simple	adjoint	$\dim G$		1
2.	$\mathrm{SL}(n, \mathbb{C}), n \geq 3$	ω_1	n	✓	0
3.	$\mathrm{SL}(n, \mathbb{C}), n \geq 3$	$2\omega_1$	$\frac{1}{2}n(n+1)$	✓	0
4.	$\mathrm{SL}(n, \mathbb{C}), n \geq 5$	ω_2	$\frac{1}{2}n(n-1)$	✓	0
5.	$\mathrm{Sp}(n, \mathbb{C}), n \geq 1$	ω_1	$2n$	✓	2
6.	$\mathrm{Sp}(n, \mathbb{C}), n \geq 3$	ω_2	$(n-1)(2n+1)$		1
7.	$\mathrm{SO}(n, \mathbb{C}), n \geq 3, n \neq 4$	ω_1	n	✓	1
8.	$\mathrm{SL}(2, \mathbb{C})$	$3\omega_1$	4	✓	2
9.	$\mathrm{SL}(6, \mathbb{C})$	ω_3	20	✓	2
10.	$\mathrm{SL}(7, \mathbb{C})$	ω_3	35	✓	0
11.	$\mathrm{SL}(8, \mathbb{C})$	ω_3	56	✓	0
12.	$\mathrm{Sp}(3, \mathbb{C})$	ω_3	14	✓	2
13.	$\mathrm{Spin}(7, \mathbb{C})$	spin	8	✓	1
14.	$\mathrm{Spin}(9, \mathbb{C})$	spin	16	✓	1
15.	$\mathrm{Spin}(10, \mathbb{C})$	half spin	16	✓	0
16.	$\mathrm{Spin}(11, \mathbb{C})$	spin	32	✓	2
17.	$\mathrm{Spin}(12, \mathbb{C})$	half spin	32	✓	2
18.	$\mathrm{Spin}(13, \mathbb{C})$	spin	64		2
19.	$\mathrm{Spin}(14, \mathbb{C})$	half spin	64	✓	0
20.	$G_2^{\mathbb{C}}$	ω_2	7	✓	1
21.	$F_4^{\mathbb{C}}$	ω_4	26		1
22.	$E_6^{\mathbb{C}}$	ω_1	27	✓	0
23.	$E_7^{\mathbb{C}}$	ω_6	56	✓	2

Table 3

ω denote the symplectic form on \mathbb{C}^{2n} fixed by $G = \mathrm{Sp}(n, \mathbb{C})$. Then

$$\beta(v_1 \wedge v_2, w_1 \wedge w_2) = \det \begin{pmatrix} \omega(v_1, w_1) & \omega(v_1, w_2) \\ \omega(v_2, w_1) & \omega(v_2, w_2) \end{pmatrix}$$

defines a G -invariant symmetric bilinear form on $V = \bigwedge_0^2 \mathbb{C}^{2n} = \ker(\omega : \bigwedge^2 \mathbb{C}^{2n} \rightarrow \mathbb{C})$. For the spin representation in case 18 the claim follows e.g. from [10, Theorem 13.1]. Finally, in case 21, it is a known fact that the representation of $F_4^{\mathbb{C}}$ on \mathbb{C}^{27} carries an invariant symmetric bilinear form (see e.g. (2.2) in [7]). \square

Remark 3.5. Let G be a simple group and H be a subgroup of type III. If G is a real form of $\mathrm{SO}(n, \mathbb{C})$, resp. $\mathrm{Sp}(n, \mathbb{C})$, then $H_{\mathbb{C}}$ must fix a symmetric, resp. skew symmetric bilinear form on V . On the other hand, if G is a real form of $\mathrm{SL}(n, \mathbb{C})$ then H cannot be maximal if it fixes a nondegenerate bilinear form, unless $H_{\mathbb{C}}$ is conjugate to $\mathrm{SO}(n, \mathbb{C})$ or (if n is even) to $\mathrm{Sp}(\frac{n}{2}, \mathbb{C})$.

It will be a consequence of the dimension bound that in most cases, a subgroup H of type III comes from a triplet in Table 3. Hence, Lemma 3.4 reduces the number of cases which must be considered for the classification of these subgroups.

4. MAXIMAL REDUCTIVE REAL SPHERICAL SUBGROUPS IN CASE $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$

We focus here on the non-split real forms $G = \mathrm{SU}(p, q)$ with $p + q = n$ and $1 \leq p \leq q$, and $G = \mathrm{SL}(m, \mathbb{H})$ with $n = 2m > 2$. Our goal is to prove the statement in Theorem 1.3 for

these groups. The dimension bounds for H are then respectively:

$$(4.1) \quad \dim H \geq 2pq - p \quad (G = \mathrm{SU}(p, q)),$$

$$(4.2) \quad \dim H \geq 2m^2 - 2m \quad (G = \mathrm{SL}(m, \mathbb{H})).$$

For later reference we record the matrix realizations of G and P . We begin with $G = \mathrm{SU}(p, q)$ which we consider as the invariance group of the Hermitian form $(\cdot, \cdot)_{p,q}$ defined by the symmetric matrix

$$J = \begin{pmatrix} 0 & I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}.$$

The Lie algebra is then given by

$$\mathfrak{su}(p, q) = \left\{ \begin{pmatrix} A & B & E \\ C & -A^* & F \\ -F^* & -E^* & D \end{pmatrix} \mid \begin{array}{l} A, B, C \in \mathrm{Mat}_{p,p}(\mathbb{C}), \\ E, F \in \mathrm{Mat}_{p,q-p}(\mathbb{C}), \\ D \in \mathrm{Mat}_{q-p,q-p}(\mathbb{C}), \\ B^*, C^*, D^* = -B, -C, -D \end{array} \right\}.$$

We choose the minimal parabolic such that

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B & E \\ 0 & -A^* & 0 \\ 0 & -E^* & D \end{pmatrix} \in \mathfrak{su}(p, q) \mid A \text{ upper triangular} \right\}$$

so that P stabilizes the isotropic flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, \dots, e_p \rangle$. Moreover we record that

$$G/P \simeq \{V_1 \subset V_2 \subset \dots \subset V_p \mid \dim_{\mathbb{C}} V_i = i, (V_p, V_p)_{p,q} = \{0\}\}$$

is the variety of full isotropic p -flags in \mathbb{C}^{p+q} . We denote by

$$\mathcal{N}_{p,q} := \{[v] \in \mathbb{P}(\mathbb{C}^n) \mid (v, v)_{p,q} = 0\}$$

the null-cone and note that there is a G -equivariant surjective map $G/P \rightarrow \mathcal{N}_{p,q}$. Moreover, if $P_{\max, \mathbb{C}}$ denotes the maximal parabolic subgroup of $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ which stabilizes $\langle e_1 \rangle$, then $P_{\mathbb{C}} \subset P_{\max, \mathbb{C}}$ and thus we have a $G_{\mathbb{C}}$ -equivariant surjection $G_{\mathbb{C}}/P_{\mathbb{C}} \rightarrow G_{\mathbb{C}}/P_{\max, \mathbb{C}} = \mathbb{P}(\mathbb{C}^n)$.

Thus we get:

Lemma 4.1. *Let $G = \mathrm{SU}(p, q)$ and $H < G$ be a real spherical subgroup. Then the following assertions hold:*

- (1) *There exists an H -orbit on \mathbb{C}^n of real co-dimension at most 3.*
- (2) *There exists an open $H_{\mathbb{C}}$ -orbit on $\mathbb{P}(\mathbb{C}^n)$, i.e. \mathbb{C}^n is a prehomogeneous vector space for $H_{\mathbb{C}}$.*

Proof. The fact that H has an open orbit on G/P implies that there is an open H -orbit in $\mathcal{N}_{p,q}$, hence the first assertion. Secondly, the fact that H has an open orbit on G/P implies that $H_{\mathbb{C}}$ has an open orbit on $G_{\mathbb{C}}/P_{\mathbb{C}}$ whence on $G/P_{\max, \mathbb{C}}$. \square

For the group $G = \mathrm{SL}(m, \mathbb{H})$ in $G_{\mathbb{C}} = \mathrm{SL}(2m, \mathbb{C})$ we choose $P < G$ the upper triangular matrices. Then

$$G/P = \{V_1 \subset V_2 \subset \dots \subset V_m = \mathbb{H}^m \mid \dim_{\mathbb{H}} V_i = i\}$$

and in particular we obtain a G -equivariant surjection $G/P \rightarrow \mathbb{P}(\mathbb{H}^m)$. Hence we get:

Lemma 4.2. *Let $H < \mathrm{SL}(m, \mathbb{H})$ be a real spherical subgroup. Then H has an open orbit on $\mathbb{P}(\mathbb{H}^m)$ and an orbit on $\mathbb{H}^m = \mathbb{C}^{2m}$ of real co-dimension at most 4.*

4.1. Exclusions of sphericity via the co-dimension bound. The criterion in Lemma 4.1 (1) is quite useful to show that many naturally occurring subgroups are not spherical. We give an application in the lemma below.

Lemma 4.3. *Let $p \geq 1$. Then $\mathrm{SO}(p-1, q)$ is not spherical in $\mathrm{SU}(p, q)$.*

Proof. Write $\mathbb{C}^{p+q} = \mathbb{C} \oplus \mathbb{C}^{p+q-1}$ and decompose vectors $v = v_1 + v_2$ accordingly. Denote by $\langle \cdot, \cdot \rangle$ the complex symmetric bilinear form on \mathbb{C}^{p+q-1} which defines $\mathrm{SO}(p-1, q)$. The following four real valued functions are H -invariant and independent:

$$\begin{aligned} f_1(v) &:= \operatorname{Re} v_1 \\ f_2(v) &:= \operatorname{Im} v_1 \\ f_3(v) &:= \operatorname{Re} \langle v_2, v_2 \rangle \\ f_4(v) &:= \operatorname{Im} \langle v_2, v_2 \rangle \end{aligned}$$

Hence each H -orbit on \mathbb{C}^{p+q} has real co-dimension at least 4, and hence H is not spherical by Lemma 4.1. \square

4.2. Type I maximal subgroups. Let $H < G$ be a maximal subgroup of type I. Then $H_{\mathbb{C}} = S(\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}))$, $n_i > 0$, is a symmetric subgroup of $G_{\mathbb{C}}$. In view of Lemma 2.2 and Berger's list [3] we thus obtain:

Lemma 4.4. *The maximal connected subgroups of Type I for $G = \mathrm{SU}(p, q)$ are given, up to conjugation, by the symmetric subgroups*

- (1) $\mathrm{S}(\mathrm{U}(p_1, q_1) \times \mathrm{U}(p_2, q_2))$ with $p_1 + p_2 = p$ and $q_1 + q_2 = q$.
- (2) $\mathrm{GL}(1, \mathbb{R}) \mathrm{SL}(p, \mathbb{C})$ if $q = p$.

Lemma 4.5. *The maximal connected subgroups of Type I for $G = \mathrm{SL}(n, \mathbb{H})$ are given, up to conjugation, by the symmetric subgroups:*

- (1) $\mathrm{S}(\mathrm{GL}(n_1, \mathbb{H}) \times \mathrm{GL}(n_2, \mathbb{H}))$ with $n_1 + n_2 = n$.
- (2) $\mathrm{U}(1) \mathrm{SL}(n, \mathbb{C})$.

4.3. Type II maximal subgroups. In this case we have $\mathbb{C}^n = \mathbb{C}^r \otimes \mathbb{C}^s$ with $2 \leq r, s$ and $H_{\mathbb{C}} = \mathrm{SL}(r, \mathbb{C}) \otimes \mathrm{SL}(s, \mathbb{C})$. In particular, $\dim H = r^2 + s^2 - 2$.

4.3.1. The case of $G = \mathrm{SU}(p, q)$.

Lemma 4.6. *Let $G = \mathrm{SU}(p, q)$ with $q \geq p \geq 2$. Then type II real spherical subgroups $H < G$ occur only for $p = q = 2$ and are given, up to conjugation, by:*

- (1) $H = \mathrm{SU}(2) \otimes \mathrm{SU}(1, 1)$,
- (2) $H = \mathrm{SU}(1, 1) \otimes \mathrm{SU}(1, 1)$.

Both cases are symmetric.

Proof. The local isomorphism $\mathrm{SU}(2, 2) \simeq \mathrm{SO}(2, 4)$ carries the subgroups (1) and (2) to $\mathrm{SO}(2, 1) \times \mathrm{SO}(3)$ and $\mathrm{SO}(1, 2) \times \mathrm{SO}(1, 2)$, respectively. Hence they are symmetric and real spherical.

Let $p + q = rs$ and $H_{\mathbb{C}} = \mathrm{SL}(r, \mathbb{C}) \otimes \mathrm{SL}(s, \mathbb{C})$. By Remark 3.2 we can exclude that H is complex, and hence we may assume that $H = H_1 \otimes H_2$ with H_1, H_2 real forms of $\mathrm{SL}(r, \mathbb{C})$ and

$\mathrm{SL}(s, \mathbb{C})$. We begin with the case where exactly one H_i , say H_1 is unitary: $H_1 = \mathrm{SU}(p_1, q_1)$. Let us first exclude the case where $H_2 = \mathrm{SL}(s, \mathbb{R})$. Note $s \geq 3$ as $\mathrm{SL}(2, \mathbb{R}) \simeq \mathrm{SU}(1, 1)$ is unitary. Then the maximal compact subgroup $K_2 := \mathrm{SO}(s, \mathbb{R})$ of H_2 acts irreducibly on \mathbb{C}^s , and hence $V = \mathbb{C}^r \otimes \mathbb{C}^s$ is irreducible for the subgroup $H_1 \otimes K_2$. Now \mathbb{C}^s carries a positive definite K_2 -invariant Hermitian form, and hence $H_1 \otimes K_2$ leaves a Hermitian form of signature $(p_1 s, q_1 s)$ invariant. By Schur's lemma this form needs to be the original form coming from $G = \mathrm{SU}(p, q)$ with signature (p, q) . It follows that the K_2 -invariant form on \mathbb{C}^s then has to be invariant under H_2 as well. This is impossible as H_2 is not compact. Likewise we can argue when $H_2 = \mathrm{SL}(k, \mathbb{H})$ which has maximal compact subgroup $K_2 = \mathrm{Sp}(k)$ acting irreducibly on \mathbb{C}^{2k} . Similar to that we can argue with both H_i either $\mathrm{SL}(\cdot, \mathbb{R})$ or $\mathrm{SL}(\cdot, \mathbb{H})$.

Finally we need to turn to the case where $H_1 = \mathrm{SU}(p_1, q_1)$ and $H_2 = \mathrm{SU}(p_2, q_2)$, with $r = p_1 + q_1$ and $s = p_2 + q_2$. We may assume that $p_1 \leq q_1$ and $p_2 \geq q_2$. Then $(p_1 - q_1)(p_2 - q_2) \leq 0$ and hence

$$p_1 p_2 + q_1 q_2 \leq p_1 q_2 + p_2 q_1.$$

We now exploit that $H_1 \otimes H_2$ leaves invariant on $\mathbb{C}^{p+q} = \mathbb{C}^r \otimes \mathbb{C}^s$ both the defining form of $\mathrm{SU}(p, q)$ and the tensor product of the defining forms of $\mathrm{SU}(p_1, q_1)$ and $\mathrm{SU}(p_2, q_2)$. By comparing signatures we thus obtain

$$(p, q) = (p_1 p_2 + q_1 q_2, p_1 q_2 + p_2 q_1).$$

With that the dimension bound (4.1) reads

$$r^2 + s^2 - 2 \geq 2(p_1 p_2 + q_1 q_2)(q_1 p_2 + q_2 p_1) - (p_1 p_2 + q_1 q_2)$$

or

$$(4.3) \quad r^2 + s^2 - 2 \geq 2p_2 q_2 (p_1^2 + q_1^2) + 2p_1 q_1 (p_2^2 + q_2^2) - (p_1 p_2 + q_1 q_2).$$

Now we distinguish various cases.

We first assume p_1, q_1, p_2, q_2 are all non-zero. If they are all 1 then we are in case (2), hence we may assume $q_1 \geq 2$ or $p_2 \geq 2$. By symmetry between r and s we can assume the latter. With $(x + y)^2 \leq 2(x^2 + y^2)$ our bound (4.3) implies

$$r^2 + s^2 - 2 \geq p_2 q_2 r^2 + p_1 q_1 s^2 - (p_1 p_2 + q_1 q_2)$$

and hence, since $p_1 q_1 \geq 1$ and $p_2 q_2 \geq 2$,

$$r^2 + s^2 - 2 \geq r^2 + s^2 + \frac{1}{2} p_2 q_2 r^2 - (p_1 p_2 + q_1 q_2).$$

As $r \geq 2$ we find

$$\frac{1}{2} p_2 q_2 r^2 \geq p_2 q_2 r = p_2 q_2 (p_1 + q_1) \geq p_2 p_1 + q_2 q_1$$

and reach a contradiction.

Hence we may assume now that $p_1 = 0$ or $q_2 = 0$. By symmetry between r and s we can assume the former, that is

$$H = \mathrm{SU}(r) \otimes \mathrm{SU}(p_2, q_2)$$

with $p_2 + q_2 = s$. The bound (4.3) now reads:

$$(4.4) \quad r^2 + s^2 - 2 \geq 2p_2 q_2 r^2 - r q_2.$$

If $s = 2$ then $p_2 = q_2 = 1$ and (4.4) gives $r^2 + 2 \geq 2r^2 - r$, from which it follows that $r = 2$ and we are in case (1).

Hence we can assume $s > 2$ and $p_2 q_2 \geq 2$. As $q_2 \leq s$ we obtain from (4.4) that $r^2 + s^2 > 4r^2 - rs$. It easily follows that $s > r$.

Now we use that H has an orbit of real co-dimension at most 3 on $\mathbb{C}^r \otimes \mathbb{C}^s$ (see Lemma 4.1). This implies that H has an orbit of real co-dimension at most 3 on $\mathbb{C}^r \otimes (\mathbb{C}^s)^* = \text{Mat}_{r,s}(\mathbb{C})$ with the action of H given as follows: $(h_1, h_2) \cdot X = h_1 X h_2^{-1}$. The map

$$(4.5) \quad \Phi : \text{Mat}_{r,s}(\mathbb{C}) \rightarrow \text{Herm}(r, \mathbb{C}), \quad X \mapsto X I_{p_2, q_2} X^*$$

is submersive and satisfies $\Phi(h_1 X h_2^{-1}) = h_1 \Phi(X) h_1^{-1}$. Hence there must be an $\text{SU}(r)$ -orbit on $\text{Herm}(r, \mathbb{C})$ of real co-dimension at most 3 and therefore $r \leq 3$ by the spectral theorem.

We are now left with the examination of the cases where $H = \text{SU}(r) \otimes \text{SU}(p_2, q_2)$ with $r = 2, 3$ and $s > r$. Set $H' := \mathbf{1} \otimes \text{SU}(p_2, q_2) < H$. Then since H has an open orbit on G/P , H' must have an orbit of codimension at most $r^2 - 1$.

To move on we introduce projective type coordinates for the flag variety G/P . We can describe a flag $\mathcal{F} \in G/P$ as follows

$$\mathcal{F} : \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \dots \subset \langle v_1, \dots, v_p \rangle$$

such that $\{v_1, \dots, v_p\}$ is orthonormal with respect to the standard Hermitian scalar product on $V = \mathbb{C}^n$. Observe in addition that all v_i are isotropic and mutually orthogonal with respect to the form $(\cdot, \cdot)_{p,q}$. It is important to note that \mathcal{F} determines the v_i uniquely up to scaling with $\text{U}(1)$.

Decompose $V = \mathbb{C}^s \oplus \dots \oplus \mathbb{C}^s$ into H' -orthogonal summands where we have two or three summands according to $r = 2$ or $r = 3$. This gives us r projections $\pi_j : \mathbb{C}^n \rightarrow \mathbb{C}^s$. Likewise for every $1 \leq m \leq p$ the π_j induce projections $\bigwedge^m \mathbb{C}^n \rightarrow \bigwedge^m \mathbb{C}^s$ which will be also denoted by π_j . Further, the invariant form $(\cdot, \cdot)_{p,q}$ induces an invariant form on $\bigwedge^m \mathbb{C}^n$, denoted by the same symbol.

We define functions g_{mjk} on G/P for $1 \leq m \leq p$ and $1 \leq j, k \leq r$ by

$$g_{mjk}(\mathcal{F}) := (\pi_j(v_1 \wedge \dots \wedge v_m), \pi_k(v_1 \wedge \dots \wedge v_m))_{p,q}.$$

Note, that for fixed m , the rational functions

$$f_{mjk} := \text{Re} \left(\frac{g_{mjk}}{g_{m11}} \right) \quad \text{and} \quad f'_{mjk} := \text{Im} \left(\frac{g_{mjk}}{g_{m11}} \right)$$

are all H' -invariant. Already for $m = 1$, we obtain $r^2 - 1$ independent functions this way. Further, as $p \geq 2$ and $n > 4$ we obtain at least one independent invariant for $m = 2$ (it will depend on the non-trivial v_2 -coordinate of \mathcal{F}), which gives a contradiction. \square

4.3.2. The case of $G = \text{SL}(m, \mathbb{H})$.

Lemma 4.7. *Let $G = \text{SL}(m, \mathbb{H})$ with $m \geq 2$. The only type II real spherical subgroup $H < G$ occurs for $m = 2$ and is given, up to conjugation, by:*

$$H = \text{SU}(1, 1) \otimes \text{SU}(2)$$

This is a symmetric subgroup.

Proof. For $H_{\mathbb{C}} = \text{SL}(r, \mathbb{C}) \otimes \text{SL}(s, \mathbb{C})$ the dimension bound (4.2) reads

$$r^2 + s^2 - 2 \geq 2m^2 - 2m.$$

The equation $rs = 2m$ together with $r, s \geq 2$ gives $r^2 + s^2 \leq m^2 + 4$. Hence $2m^2 - 2m \leq m^2 + 4$, which implies $m = 2$. Then $r = s = 2$. The local isomorphism $\text{SL}(2, \mathbb{H}) \simeq \text{SO}(1, 5)$

carries $H = \mathrm{SU}(1, 1) \otimes \mathrm{SU}(2)$ to $\mathrm{SO}(1, 2) \times \mathrm{SO}(3)$, which is symmetric. On the other hand, $H = \mathrm{SU}(1, 1) \otimes \mathrm{SU}(1, 1)$ is excluded by the rank inequality. \square

4.4. Type III maximal reductive subgroups. Here $H_{\mathbb{C}}$ is simple and acts irreducibly on \mathbb{C}^n .

4.4.1. *The case of $G = \mathrm{SU}(p, q)$.*

Lemma 4.8. *Let $p + q \geq 3$ and let $H < \mathrm{SU}(p, q)$ be a reductive real spherical subgroup of type III. Then, up to conjugation, H is one of the following symmetric subgroups:*

- (1) $\mathrm{SO}(p, q)$.
- (2) $\mathrm{SO}^*(2p)$ if $p = q$.
- (3) $\mathrm{Sp}(p/2, q/2)$ if p, q are even.
- (4) $\mathrm{Sp}(p, \mathbb{R})$ if $p = q$.

Proof. According to [17], the assertion is true for $p = 1$ and henceforth we assume that $q \geq p \geq 2$. By the dimension bound (4.1) we have for $n = p + q$

$$(4.6) \quad \dim H \geq 2pq - p = 2p(n - p) - p \geq 4n - 10,$$

as the right hand side is an increasing function of p . We recall from Lemma 4.1 that $V = \mathbb{C}^n$ is a prehomogeneous vector space for $H_{\mathbb{C}}$, and since V is irreducible and $H_{\mathbb{C}}$ is simple, we can apply Proposition 3.3 and Lemma 3.4 as explained in Remark 3.5.

- $H_{\mathbb{C}} = \mathrm{SL}(m, \mathbb{C})$ acting on the space $V = \mathrm{Skew}(m, \mathbb{C})$ of skew symmetric matrices, $m \geq 5$. Here $n = \frac{1}{2}m(m - 1)$ and $\dim H = m^2 - 1$. Hence by (4.6) we obtain $m^2 - 2m - 9 \leq 0$ which is excluded for $m \geq 5$.

- $H_{\mathbb{C}} = \mathrm{SL}(m, \mathbb{C})$ acting on the space $V = \mathrm{Sym}(m, \mathbb{C})$ of symmetric matrices, $m \geq 3$. Here $n = \frac{1}{2}m(m + 1)$ and we get $m^2 + 2m - 9 \leq 0$, which is excluded with $m \geq 3$.

- $H_{\mathbb{C}} = \mathrm{SO}(n, \mathbb{C})$ acting on $V = \mathbb{C}^n$. This leads to (1) and (2).

- $H_{\mathbb{C}} = \mathrm{Sp}(m, \mathbb{C})$ acting on $\mathbb{C}^n = \mathbb{C}^{2m}$. This leads to (3) and (4).

- *The sporadic prehomogeneous vector spaces.* Since we assume that $p \geq 2$, the dimension bound gives no possibilities. \square

4.4.2. *The case of $G = \mathrm{SL}(m, \mathbb{H})$.*

Lemma 4.9. *Let $H < \mathrm{SL}(m, \mathbb{H})$ for $m \geq 3$ be a real spherical subgroup of type III. Then H is conjugate to one of the following symmetric subgroups:*

- (1) $\mathrm{SO}^*(2m)$
- (2) $\mathrm{Sp}(p, q)$, $p + q = m$.

Proof. Let $V = \mathbb{C}^{2m} = \mathbb{C}^n$. By (4.2) a spherical subgroup H satisfies

$$\dim H \geq 2m^2 - 2m > 2m = n = \dim_{\mathbb{C}} V,$$

as $m > 2$. Since $H_{\mathbb{C}}$ acts via ρ irreducibly on V , it follows from Lemma 4.2 that the triplet $(H_{\mathbb{C}}, \rho, V)$ appears among the even-dimensional cases in Proposition 3.3. Further, we can ignore some cases due to Lemma 3.4, as explained in Remark 3.5.

- $H_{\mathbb{C}} = \mathrm{SL}(k, \mathbb{C})$, acting on $V = \mathrm{Skew}(k, \mathbb{C})$ with $k \geq 5$. The dimension bound for H reads
(4.7)
$$k^2 - 1 \geq 2m^2 - 2m.$$

Since $2m = \frac{1}{2}k(k-1)$ and $k \geq 5$, we have $m \geq k \geq 5$. Furthermore, $k^2 = 4m + k$ and by (4.7) we get the contradiction

$$4m + k - 1 \geq 2m(m-1) \geq 8m.$$

- $H_{\mathbb{C}} = \mathrm{SL}(k, \mathbb{C})$, acting on $V = \mathrm{Sym}(k, \mathbb{C})$ with $k \geq 3$. Here $2m = \frac{1}{2}k(k+1)$. Since $k \geq 3$, we have $m \geq k \geq 3$. Furthermore, $k^2 = 4m - k$ and by (4.7) we get the contradiction

$$4m - k - 1 \geq 2m(m-1) \geq 4m.$$

- $H_{\mathbb{C}} = \mathrm{SO}(2m, \mathbb{C})$ acting on $V = \mathbb{C}^{2m}$. The real form $H = \mathrm{SO}^*(2m)$ gives case (1) of the lemma. The real form $H = \mathrm{SO}(p, q)$, $p + q = 2m$ cannot occur, since its maximal compact subgroup $\mathrm{SO}(p) \times \mathrm{SO}(q)$ must be conjugate to a subgroup of $K = \mathrm{Sp}(m) \subseteq \mathrm{SU}(m, m)$ from which we conclude $p = q = m$. But then, $\mathrm{rank}(H) = m > m-1 = \mathrm{rank}(G)$.

- $H_{\mathbb{C}} = \mathrm{Sp}(m, \mathbb{C})$ acting on $V = \mathbb{C}^{2m}$. The real form $H = \mathrm{Sp}(p, q)$ with $p + q = m$ gives case (2) of the lemma. The real form $H = \mathrm{Sp}(m, \mathbb{R})$ does not occur, since its rank equals $2m$ which is greater than $\mathrm{rank}(G) = m-1$.

- $H_{\mathbb{C}} = \mathrm{SL}(k, \mathbb{C})$ acting on $V = \bigwedge^3 \mathbb{C}^k$, $k = 7, 8$. It is easy to see that for $k = 8$ the dimension bound is violated, while for $k = 7$ the dimension of V is odd.

- $H_{\mathbb{C}} = \mathrm{Spin}(k, \mathbb{C})$ acting on a half spin representation, $k = 10, 14$. The representation spaces are \mathbb{C}^{16} and \mathbb{C}^{64} respectively. The dimension bound for H reads $\frac{1}{2}k(k-1) \geq 2m(m-1)$, whence we get the contradiction $k \geq 2m$.

- $H_{\mathbb{C}}$ of exceptional type: Lemma 3.4 excludes all cases except $E_6^{\mathbb{C}}$ but in this case $\dim V$ is odd. \square

This concludes the proof of Theorem 1.3 for $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$.

5. MAXIMAL REDUCTIVE REAL SPHERICAL SUBGROUPS FOR THE ORTHOGONAL GROUPS

In this chapter we let $G_{\mathbb{C}} = \mathrm{SO}(n, \mathbb{C})$ and our focus is on the two real forms $G = \mathrm{SO}(p, q)$, $p + q = n$ and $p \leq q$ and $G = \mathrm{SO}^*(2m)$ with $n = 2m$. We assume throughout that $n \geq 5$, and our goal is then to prove the statement in Theorem 1.3 for these groups.

The dimension bounds for H are as follows:

$$(5.1) \quad \dim H \geq pq - p \quad (G = \mathrm{SO}(p, q)),$$

$$(5.2) \quad \dim H \geq m^2 - \frac{3}{2}m \quad (G = \mathrm{SO}^*(2m)).$$

For further reference we record the matrix realizations of G and P . We begin with $G = \mathrm{SO}(p, q)$ which we consider as the invariance group of the symmetric form $\langle \cdot, \cdot \rangle_{p,q}$ defined by

$$\begin{pmatrix} 0 & I_p & 0 \\ I_p & 0 & 0 \\ 0 & 0 & I_{q-p} \end{pmatrix}.$$

Accordingly we obtain for the Lie algebra

$$\mathfrak{so}(p, q) = \left\{ \begin{pmatrix} A & B & E \\ C & -A^T & F \\ -F^T & -E^T & D \end{pmatrix} \mid \begin{array}{l} A, B, C \in \text{Mat}_{p,p}(\mathbb{R}), \\ E, F \in \text{Mat}_{p,q-p}(\mathbb{R}), \\ D \in \text{Mat}_{q-p,q-p}(\mathbb{C}), \\ B^T, C^T, D^T = -B, -C, -D \end{array} \right\}.$$

We choose the minimal parabolic such that

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B & E \\ 0 & -A^T & 0 \\ 0 & -E^T & D \end{pmatrix} \in \mathfrak{so}(p, q) \mid A \text{ upper triangular} \right\}$$

so that P stabilizes the isotropic real flag $\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \dots \subset \langle e_1, \dots, e_p \rangle$ in \mathbb{R}^n . Moreover

$$G/P \simeq \{V_1 \subset \dots \subset V_p \mid V_p \subset \mathbb{R}^n, \dim_{\mathbb{R}} V_i = i, \langle V_p, V_p \rangle_{p,q} = \{0\}\}$$

is the variety of full isotropic p -flags in \mathbb{R}^{p+q} with respect to the symmetric bilinear form $\langle \cdot, \cdot \rangle_{p,q}$.

Let us denote

$$\mathcal{N}_{p,q}^{\mathbb{R}} := \{[v] \in \mathbb{P}(\mathbb{R}^n) \mid \langle v, v \rangle_{p,q} = 0\}$$

and note that there is a G -equivariant surjective map $G/P \rightarrow \mathcal{N}_{p,q}^{\mathbb{R}}$. Hence we obtain the following lemma.

Lemma 5.1. *Let $G = \text{SO}(p, q)$ and $H < G$ a real spherical subgroup. Then there exists an H -orbit on \mathbb{R}^n of co-dimension at most 2.*

Proof. The fact that H has an open orbit on G/P implies that there is an open H -orbit in $\mathcal{N}_{p,q}^{\mathbb{R}}$. \square

We continue to recall a few structural facts for the group $\text{SO}^*(2m)$. We identify \mathbb{H}^m with \mathbb{C}^{2m} via $\mathbb{H}^m = \mathbb{C}^m \oplus j\mathbb{C}^m$. Denote by $h \mapsto \bar{h}$ the

conjugation on \mathbb{H}^m . The group $\text{SO}^*(2m)$ consists of the right \mathbb{H} -linear transformations on \mathbb{H}^m which preserve the \mathbb{H} -valued form

$$\phi(h, h') = \bar{h}_1 j h'_1 + \dots + \bar{h}_m j h'_m \quad (h_i, h'_i \in \mathbb{H}).$$

Observe that ϕ is a so-called skew-Hermitian form, i.e. it is sesquilinear and skew. We recall that *sesquilinear* means $\phi(hx, h'x') = \bar{x}\phi(h, h')x'$ for all $h, h' \in \mathbb{H}^m$, $x, x' \in \mathbb{H}$, and *skew* refers to $\overline{\phi(h, h')} = -\phi(h', h)$.

Denote the \mathbb{C} -part of $\phi(h, h')$ by $(h, h')_{m,m} \in \mathbb{C}$ and the $j\mathbb{C}$ -part by $\langle h, h' \rangle \in \mathbb{C}$. If we write elements $h \in \mathbb{H}^m$ as $h = x + jy$ with $x, y \in \mathbb{C}^m$, then

$$(h, h')_{m,m} = \bar{y}^T x' - \bar{x}^T y'$$

and

$$\langle h, h' \rangle = x^T x' + y^T y'.$$

Notice that $i(\cdot, \cdot)_{m,m}$ is a Hermitian form of signature (m, m) . In particular, if we view $\text{SO}^*(2m)$ as a subgroup of $\text{SL}(2m, \mathbb{C})$, then $\text{SO}^*(2m) = \text{SO}(2m, \mathbb{C}) \cap \text{SU}(m, m)$.

The minimal flag variety is given by isotropic right \mathbb{H} -flags

$$(5.3) \quad G/P = \{V_1 \subset \dots \subset V_{[m/2]} \subset \mathbb{H}^m \mid \phi(V_i, V_i) = \{0\}, \dim_{\mathbb{H}} V_i = i\}.$$

Remark 5.2. Observe that the sesquilinear form ϕ is uniquely determined by its \mathbb{C} -part or $j\mathbb{C}$ -part. Hence an \mathbb{H} -subspace $V_i \subset \mathbb{H}^m$ is isotropic if and only if it is isotropic for $\langle \cdot, \cdot \rangle$ (or $(\cdot, \cdot)_{m,m}$). Recall that $G_{\mathbb{C}}/B_{\mathbb{C}}$ is the variety of $\langle \cdot, \cdot \rangle$ -isotropic (left) \mathbb{C} -flags in $\mathbb{C}^{2m} = \mathbb{H}^m$. Hence the right hand side of (5.3) embeds totally real into the quotient of $G_{\mathbb{C}}/B_{\mathbb{C}}$ consisting of even-dimensional isotropic complex flags. A simple dimension count then shows equality in (5.3).

5.1. Type I maximal subgroups. Let $H < G$ be a maximal subgroup of type I. Then $H_{\mathbb{C}} = S(\mathrm{GL}(n_1, \mathbb{C}) \times \mathrm{GL}(n_2, \mathbb{C}))$, $n_i > 0$ and $n = n_1 + n_2$, or $H_{\mathbb{C}} = \mathrm{GL}(n/2, \mathbb{C})$ for n even. In both cases $H_{\mathbb{C}}$ is a symmetric subgroup of $G_{\mathbb{C}}$. Hence with Lemma 2.2 and Berger's list [3] we obtain:

Lemma 5.3. *Let $H < \mathrm{SO}^*(2m)$ be a subgroup of type I. Then H is symmetric and up to conjugation one of the following:*

- (1) $\mathrm{SO}^*(2m_1) \times \mathrm{SO}^*(2m_2)$ with $m_1 + m_2 = m$, $m_1, m_2 > 0$,
- (2) $\mathrm{SO}(m, \mathbb{C})$,
- (3) $\mathrm{GL}(m/2, \mathbb{H})$ for m even,
- (4) $\mathrm{U}(k, l)$ with $k + l = m$.

Lemma 5.4. *Let $H < \mathrm{SO}(p, q)$ be a subgroup of type I. Then H is symmetric and up to conjugation one of the following:*

- (1) $\mathrm{SO}(p_1, q_1) \times \mathrm{SO}(p_2, q_2)$ with $p_1 + p_2 = p$ and $q_1 + q_2 = q$,
- (2) $\mathrm{SO}(p, \mathbb{C})$ for $p = q$,
- (3) $\mathrm{GL}(p, \mathbb{R})$ with $p = q$.
- (4) $\mathrm{U}(p/2, q/2)$ for p, q even.

5.2. Type II maximal reductive subgroups. Here we suppose that $H_{\mathbb{C}}$ is maximal reductive subgroup of $G_{\mathbb{C}} = \mathrm{SO}(n, \mathbb{C})$, $n \geq 5$ of type II. Hence there are the two possibilities:

- $H_{\mathbb{C}} = \mathrm{SO}(r, \mathbb{C}) \otimes \mathrm{SO}(s, \mathbb{C})$ with $rs = n$, $3 \leq r \leq s$, and $r, s \neq 4$.
- $H_{\mathbb{C}} = \mathrm{Sp}(r, \mathbb{C}) \otimes \mathrm{Sp}(s, \mathbb{C})$ with $4rs = n$ and $1 \leq r \leq s$.

5.2.1. The case of $G = \mathrm{SO}^*(2m)$. Note that $\mathfrak{so}^*(6) \simeq \mathfrak{su}(1, 3)$ and $\mathfrak{so}^*(8) \simeq \mathfrak{so}(2, 6)$ are treated elsewhere, so that we may assume that $m \geq 5$ in the sequel.

Lemma 5.5. *Let $G = \mathrm{SO}^*(2m)$ for $m \geq 5$. Then there exist no real spherical subgroups $H < G$ of type II.*

Proof. Because if $m \geq 5$ no type II subgroup satisfies the dimension bound. □

5.2.2. The case of $G = \mathrm{SO}(p, q)$.

Lemma 5.6. *Let $G = \mathrm{SO}(p, q)$ with $p + q \geq 5$. Then type II real spherical subgroups $H < G$ occur only for $p = q = 4$ and are given, up to conjugation, by:*

- (1) $H = \mathrm{Sp}(1, \mathbb{R}) \otimes \mathrm{Sp}(2, \mathbb{R})$,
- (2) $H = \mathrm{Sp}(1) \otimes \mathrm{Sp}(1, 1)$.

Both cases are symmetric.

Proof. We first prove that the groups listed under (1) and (2) are symmetric and hence real spherical. Write $H = H_1 \otimes H_2$ and $\mathbb{C}^8 = \mathbb{C}^2 \otimes \mathbb{C}^4$. The symplectic forms Ω_i on \mathbb{C}^{2i} defined by H_i give rise to the $\mathrm{SO}(4, 4)$ -invariant symmetric form $\langle \cdot, \cdot \rangle = \Omega_1 \otimes \Omega_2$ on \mathbb{C}^8 . Write J_1 and

J_2 for the matrices defining Ω_1 and Ω_2 . Then $B = \Omega_1(J_1 \cdot, \cdot) \otimes \Omega_2(J_2 \cdot, \cdot)$ defines a symmetric bilinear form on \mathbb{C}^8 and we write $g \mapsto g^t$ for the corresponding transpose on matrices. Then the assignment $g \mapsto (J_1 \otimes J_2)g^{-t}(J_1 \otimes J_2)$ defines an involution on $G = \mathrm{SO}(4, 4)$ with fixed group H . Hence H is symmetric (and outer isomorphic to $\mathrm{SO}(2, 1) \times \mathrm{SO}(2, 3)$, respectively $\mathrm{SO}(3) \times \mathrm{SO}(1, 4)$, from Berger's list).

Let $H = H_1 \otimes H_2$ be a type II subgroup. We consider first the case where each $H_{i\mathbb{C}}$ is symplectic. We start with $H = \mathrm{Sp}(r, \mathbb{R}) \otimes \mathrm{Sp}(s, \mathbb{R})$. The invariant Hermitian form on each factor gives a an invariant Hermitian form on the tensor product with signature $(2rs, 2rs)$ which then must be equal to (p, q) . The dimension bound (5.1) becomes

$$r(2r + 1) + s(2s + 1) \geq 4r^2s^2 - 2rs.$$

For $r \geq 1$ and $s \geq 2$ we have $r(2r + 1) \leq 3r^2 \leq \frac{3}{4}r^2s^2$ and $s(2s + 1) \leq \frac{5}{2}r^2s^2$. It follows that $4r^2s^2 - 2rs \leq \frac{13}{4}r^2s^2$ which easily implies $rs < 3$. Since $4rs = n \geq 5$ it follows that $r = 1$ and $s = 2$. These data produce the first symmetric subgroup mentioned in the lemma.

For $H = \mathrm{Sp}(r, \mathbb{R}) \otimes \mathrm{Sp}(p_2, q_2)$ we obtain the same signature condition $p = q = 2rs$ as before and hence $H = \mathrm{Sp}(1, \mathbb{R}) \otimes \mathrm{Sp}(1, 1)$. Up to an outer automorphism this is a real form of a symmetric subgroup in $G_{\mathbb{C}}$, which can be excluded with Berger's list for $G = \mathrm{SO}(4, 4)$.

The case where $H = \mathrm{Sp}(p_1, q_1) \otimes \mathrm{Sp}(p_2, q_2)$ with $r = p_1 + q_1$ and $s = p_2 + q_2$ is treated analogously as Lemma 4.6. We can assume $p_1 \leq q_1$ and $p_2 \geq q_2$. The group H leaves invariant a Hermitian form of signature $(4(p_1p_2 + q_1q_2), 4(p_1q_2 + p_2q_1))$, which must then equal (p, q) . Then the dimension bound

$$r(2r + 1) + s(2s + 1) \geq 16p_2q_2(p_1^2 + q_1^2) + 16p_1q_1(p_2^2 + q_2^2) - 4(p_1p_2 + q_1q_2)$$

leads to the absurd unless $p_1 = 0$ and $H = \mathrm{Sp}(r) \otimes \mathrm{Sp}(p_2, q_2)$ with $r \leq s$. Using a matrix submersion as (4.5) we obtain with Lemma 5.1 that $r = 1$, hence $H = \mathrm{Sp}(1) \otimes \mathrm{Sp}(p_2, q_2)$ and $G = \mathrm{SO}(4p_2, 4q_2)$. Set $H' := \mathrm{Sp}(p_2, q_2) < H$. Then H' is of co-dimension 3 in H and thus H' admits an orbit of co-dimension 3 on G/P . We parametrize flags $\mathcal{F} \in G/P$ as in Lemma 4.6. Let $V = \mathbb{R}^{4p_2+4q_2} \simeq \mathbb{C}^{2p_2+2q_2}$. First we note that there are three independent real symplectic forms which are invariant under H' . In fact, if Ω is the complex symplectic form on $\mathbb{C}^{2p_2+2q_2}$ which defines H' , then $\Omega_1 = \mathrm{Re} \Omega$, and $\Omega_2 = \mathrm{Im} \Omega$ give two independent symplectic forms. A third form is given by $\Omega_3 = \mathrm{Im}(\cdot, \cdot)_{2p_2, 2q_2}$. Concretely, the Ω_i are given as follows: Out of the standard symplectic forms J_i on \mathbb{R}^4

$$J_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

we build the forms $\mathbf{J}_i = \mathrm{diag}(J_i) \in \mathrm{Mat}_{p+q}(\mathbb{R})$. Then $\Omega_i(\cdot, \cdot) = \langle \mathbf{J}_i \cdot, \cdot \rangle_{p, q}$.

This gives us two independent rational invariants

$$f_i(\mathcal{F}) := \frac{\Omega_i(v_1 \wedge v_2)}{\Omega_3(v_1 \wedge v_2)} \quad (i = 1, 2).$$

Further invariants we obtain via

$$g_{j,k}(\mathcal{F}) = \frac{(\Omega_j \wedge \Omega_k)(v_1 \wedge v_2 \wedge v_3 \wedge v_4)}{(\Omega_3 \wedge \Omega_3)(v_1 \wedge v_2 \wedge v_3 \wedge v_4)} \quad (1 \leq j \leq k \leq 3, (j, k) \neq (3, 3)).$$

Clearly each $g_{j,k}$ is independent to $\{f_1, f_2\}$ as the f_i only depend on the first two coordinates v_1, v_2 . Moreover, if $p_2 > 1$ we obtain additional invariants with an analogous construction on $\bigwedge^6 V$. Thus for $p_2 > 1$ we obtain at least 4 algebraically independent H' -invariants on G/P contradicting the fact that the generic H' -orbit is of co-dimension at most 3. This leaves us to investigate the case with $p_2 = 1$. Now if $q_2 = 1$, the $g_{j,k}$ are all dependent and H is the second symmetric subgroup mentioned in the lemma. If $q_2 > 1$, then we obtain at least 4 algebraically independent functions out of f_i, g_{jk} . To verify that we may restrict ourselves to the case $q_2 = 2$. We fix the first two coordinates of \mathcal{F} to be $v_1 = e_1, v_2 = e_6$. For $\lambda, \mu, \nu, \epsilon, \delta \in \mathbb{R}$ we consider

$$\tilde{v}_3 = e_3 + \lambda e_4 + \mu e_7 + \nu e_8 + \epsilon e_9 \quad \tilde{v}_4 = e_3 - \mu e_7 + \delta e_{10}.$$

Then $\{v_1, v_2, \tilde{v}_3, \tilde{v}_4\}$ is a set of mutually orthogonal vectors with respect to $\langle \cdot, \cdot \rangle_{p,q}$. Moreover \tilde{v}_3 , resp. \tilde{v}_4 , is isotropic provided that $2(\mu + \lambda\nu) + \epsilon^2 = 0$, resp. $-2\mu + \delta^2 = 0$. We choose now the parameters such that both \tilde{v}_3 and \tilde{v}_4 are isotropic. Let v_3, v_4 be unit vectors obtained from \tilde{v}_3, \tilde{v}_4 .

This then gives us isotropic flags

$$\mathcal{F} = \{\langle e_1 \rangle \subset \langle e_1, e_6 \rangle \subset \langle e_1, e_6, v_3 \rangle \subset \langle e_1, e_6, v_3, v_4 \rangle\}.$$

Then

$$g_{11}(\mathcal{F}) = \frac{\Omega_1(v_1, v_4)\Omega_1(v_3, v_2)}{\Omega_3(v_1, v_2)\Omega_3(v_3, v_4)} \quad \text{and} \quad g_{22}(\mathcal{F}) = \frac{\Omega_2(v_1, v_3)\Omega_2(v_2, v_4)}{\Omega_3(v_1, v_2)\Omega_3(v_3, v_4)}$$

and in particular

$$g_{11}(\mathcal{F}) = \frac{\lambda\mu}{\mu\lambda - \nu + \epsilon\delta} \quad \text{and} \quad g_{22}(\mathcal{F}) = \frac{-\nu}{\mu\lambda - \nu + \epsilon\delta}.$$

It follows that $\{f_1, f_2, g_{11}, g_{22}\}$ are independent and hence H is not real spherical for $q_2 > 1$.

Next we look at the case where $H = H_1 \otimes H_2$ with both complexifications orthogonal. We begin with both $H_i = \text{SO}^*(2m_i)$ quaternionic, and $m_i > 2$. The invariant Hermitian form on each factor gives a an invariant Hermitian form on the tensor product with signature $(2m_1m_2, 2m_1m_2)$ which then must be equal to (p, q) . Then the dimension bound

$$2m_1^2 - m_1 + 2m_2^2 - m_2 \geq 4m_1^2m_2^2 - 2m_1m_2$$

is easily seen to be violated. Similarly if $H_1 = \text{SO}^*(2m_1)$ and $H_2 = \text{SO}(p_2, q_2)$ with $p_2 + q_2 = s$, then $p = q = m_1(p_2 + q_2)$ by a signature argument, and exactly the same bound as above results.

This reduces to the final case where $H = \text{SO}(p_1, q_1) \otimes \text{SO}(p_2, q_2)$, which is treated similarly as the previous case of $H = \text{Sp}(p_1, q_1) \otimes \text{Sp}(p_2, q_2)$. Comparing signatures we find that $p = p_1p_2 + q_1q_2$ and $q = p_1q_2 + p_2q_1$, and the dimension bound then implies $H = \text{SO}(r) \otimes \text{SO}(p_2, q_2)$ with $r \leq s$. By applying a matrix submersion as (4.5) we obtain with Lemma 5.1 that H must have an orbit on $\text{Sym}(r, \mathbb{R})$ of codimension 2. This contradicts that $r \geq 3$. \square

5.3. Type III maximal subgroups. We assume that $H_{\mathbb{C}}$ is simple and acts irreducibly on V .

5.3.1. The case $G = \mathrm{SO}(p, q)$.

Lemma 5.7. *Let $G = \mathrm{SO}(p, q)$ for $1 \leq p \leq q$ and $p + q \geq 5$. Then the only real spherical subgroups $H < G$ of type III are given, up to isomorphism, by*

- (1) $\mathfrak{h} = \mathbf{G}_2^1$ in $\mathfrak{g} = \mathfrak{so}(3, 4)$.
- (2) $\mathfrak{h} = \mathfrak{spin}(3, 4)$ in $\mathfrak{g} = \mathfrak{so}(4, 4)$ (two conjugacy classes swapped by an outer automorphism of \mathfrak{g}).

These pairs are absolutely spherical and the second one is symmetric.

Proof. For $p = 1$ it follows from [17] that there are no such spherical subgroups. The case $\mathfrak{so}(2, 3)$ is quasi-split and features no type III subalgebras according to Krämer. Hence we may assume here $2 \leq p \leq q$, $q \geq 3$ and $p + q > 5$. Then

$$(pq - p) - (p + q) = (p - 2)(q - 3) + p + q - 6 \geq 0.$$

Hence the dimension bound (5.1) implies

$$(5.4) \quad \dim H \geq pq - p \geq p + q = \dim V.$$

In particular, we can apply Proposition 3.3 and Remark 3.5. We observe also that $pq - p > p + q$ if $p + q > 6$.

- $H_{\mathbb{C}}$, *adjoint representation*. Then $\dim H = \dim V$, which is excluded unless $\dim H = 6$, by the strictness of (5.4). Then $H = \mathrm{SO}(3, 1)$ and $H_{\mathbb{C}}$ is not simple.

- $H_{\mathbb{C}} = \mathrm{SO}(m, \mathbb{C})$ *acting on* $V = \mathbb{C}^m$. This is possible, but then we would have $H_{\mathbb{C}} = G_{\mathbb{C}}$.

- $H_{\mathbb{C}} = \mathrm{Sp}(m, \mathbb{C})$ *acting on* $\bigwedge_0^2 \mathbb{C}^{2m}$ for $m \geq 3$. Here $n = 2m^2 - m - 1 = p + q$. The dimension bound then gives $2m^2 + m \geq p(2m^2 - m - 2 - p)$. Already for $p = 2$ this implies $m \leq 2$, and hence there are no solutions.

- $H_{\mathbb{C}} = \mathrm{Spin}(m, \mathbb{C})$ *acting on a spin representation*, $m = 7, 9$. Since the representation spaces are \mathbb{C}^8 and \mathbb{C}^{16} respectively, the dimension bound leaves the following possibilities:

- $G = \mathrm{SO}(2, 6)$, $\mathrm{SO}(3, 5)$ or $\mathrm{SO}(4, 4)$ if $m = 7$,
- $G = \mathrm{SO}(2, 14)$ or $\mathrm{SO}(3, 13)$ if $m = 9$.

It follows from the signature laws of the (half-)spin representations [10, Theorems 13.1 and 13.8] that only equal parity signatures can occur. Hence only $G = \mathrm{SO}(4, 4)$ is possible. In that case $\mathfrak{h} = \mathfrak{spin}(3, 4) := \mathfrak{spin}(7, \mathbb{C}) \cap \mathfrak{so}(4, 4)$. It is symmetric by Lemma 2.2, since $\mathfrak{spin}(7, \mathbb{C})$ is symmetric in $\mathfrak{so}(8, \mathbb{C})$.

- $H_{\mathbb{C}}$ *of exceptional type*. The case $H_{\mathbb{C}} = \mathbf{G}_2^{\mathbb{C}}$ is possible; with $\mathfrak{h} = \mathbf{G}_2^1$, the pair $(\mathfrak{so}(3, 4), \mathfrak{h})$ is absolutely spherical (see Table 7). In view of Lemma 3.4 we are left with $H_{\mathbb{C}} = \mathbf{F}_4^{\mathbb{C}}$. Then $\dim V = 26$ and the dimension bound implies that $G = \mathrm{SO}(2, 24)$. The only non-compact real form of $H_{\mathbb{C}}$ with rank ≤ 2 is \mathbf{F}_4^2 . Its representation space

$$V = \{X \in \mathrm{Herm}(3, \mathbb{O})_{\mathbb{C}} : \mathrm{Tr} X = 0\}.$$

According to (2.2) in [7], the space V carries an invariant symmetric bilinear form with signature $(10, 16)$. This is different from $(2, 24)$. \square

5.3.2. The case $G = \mathrm{SO}^*(2m)$.

Lemma 5.8. *Let $G = \mathrm{SO}^*(2m)$ for $m \geq 5$. Then there exists no real spherical subgroup $H < G$ of type III.*

Proof. By assumption $m \geq 2$ and hence

$$m^2 - \frac{3}{2}m = \frac{m}{2} \cdot (2m - 3) > 2m = \dim(V).$$

Hence, if H is spherical it follows from (5.2) that $\dim H > \dim V$. In particular, V is then a representation from Proposition 3.3, to which also Remark 3.5 applies.

- $H_{\mathbb{C}}$ simple, adjoint representation. Since $\dim H = \dim V$, this is impossible by the strictness of the dimension bound.

- $H_{\mathbb{C}} = \mathrm{SO}(k, \mathbb{C})$ acting on $V = \mathbb{C}^k$. This is possible for $k = 2m$, but then we would have $H_{\mathbb{C}} = G_{\mathbb{C}}$.

- $H_{\mathbb{C}} = \mathrm{Spin}(k, \mathbb{C})$ acting on a (half-)spin representation for $k = 7, 9$. Note that $k = 7$ is excluded since $m \geq 5$. For $\mathrm{Spin}(9, \mathbb{C})$ on \mathbb{C}^{16} we have $m = 8$ and $\dim H = 36 < 52$, so H does not satisfy the dimension bound.

- $H_{\mathbb{C}} = \mathrm{Sp}(k, \mathbb{C})$ acting on $\bigwedge_0^2 \mathbb{C}^{2k}$. Here $2m = 2k^2 - k - 1$. Since $k \geq 3$, we have $m \geq 7$. Hence, it follows from

$$\dim H = 2k^2 + k = 2m + 2k + 1 \geq m(m - \frac{3}{2})$$

and $k \leq m$ that $4m + 1 \geq \frac{11}{2}m$, which is impossible.

- $H_{\mathbb{C}}$ of exceptional type. Only for $F_4^{\mathbb{C}}$ is $\dim V$ even and not excluded by Lemma 3.4. Then $m = 13$ and (5.2) is invalid. \square

This concludes the proof of Theorem 1.3 for $G_{\mathbb{C}} = \mathrm{SO}(n, \mathbb{C})$.

6. MAXIMAL REDUCTIVE REAL SPHERICAL SUBGROUPS FOR THE SYMPLECTIC GROUPS

We only consider the real forms $G = \mathrm{Sp}(p, q)$ of $G_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C})$, $p + q = n$ and $0 < p \leq q$, as the real form $\mathrm{Sp}(n, \mathbb{R})$ is split. Then $\mathrm{rank}_{\mathbb{R}} G = p$ and the dimension bound is

$$(6.1) \quad \dim H \geq 4pq - p.$$

For later reference we record some structural facts for the group $\mathrm{Sp}(p, q)$. As before we identify \mathbb{H}^n with \mathbb{C}^{2n} and denote by $h \mapsto \bar{h}$ the conjugation on \mathbb{H}^n . The group $\mathrm{Sp}(p, q)$ consists of the right \mathbb{H} -linear transformations on \mathbb{H}^n which preserve the Hermitian \mathbb{H} -valued form

$$\phi(h, h') = \bar{h}_1 h'_1 + \dots + \bar{h}_p h'_p - \bar{h}_{p+1} h'_{p+1} - \dots - \bar{h}_n h'_n.$$

Similar to the $\mathrm{SO}^*(2m)$ -case the \mathbb{C} -part of ϕ yields a Hermitian form $(\cdot, \cdot)_{2p, 2q}$ and the $j\mathbb{C}$ -part a symplectic form $\langle \cdot, \cdot \rangle$, both being kept invariant under $\mathrm{Sp}(p, q)$. In particular, if we view $\mathrm{Sp}(p, q)$ as a subgroup of $\mathrm{SL}(2n, \mathbb{C})$, then $\mathrm{Sp}(p, q) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{SU}(2p, 2q)$.

The minimal flag variety is given by the isotropic right \mathbb{H} -flags:

$$(6.2) \quad G/P = \{V_1 \subset \dots \subset V_p \subset \mathbb{H}^n \mid \dim_{\mathbb{H}} V_i = i, \phi(V_i, V_i) = \{0\}\}.$$

Lemma 6.1. *Let $H < \mathrm{Sp}(p, q)$ be a real spherical subgroup. Then H admits an orbit on $\mathbb{P}(\mathbb{H}^n)$ of real codimension at most 1 and an orbit on $\mathbb{H}^n = \mathbb{C}^{2p+2q}$ of real co-dimension at most 5.*

Proof. Let \mathcal{L} be the variety of ϕ -isotropic \mathbb{H} -lines. According to (6.2) \mathcal{L} is a G -quotient of G/P and hence a real spherical subgroup $H < G$ must admit an open orbit on \mathcal{L} . Observe that a line $v\mathbb{H}$ is isotropic if and only if the real valued function $v \mapsto \phi(v, v)$ vanishes. From that the assertion follows. \square

6.1. Type I maximal subgroups. Let $H < G$ be a maximal subgroup of type I. Then $H_{\mathbb{C}} = \mathrm{Sp}(r, \mathbb{C}) \times \mathrm{Sp}(s, \mathbb{C})$, $r, s > 0$ and $n = r + s$ or $H_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$. In both cases $H_{\mathbb{C}}$ is a symmetric subgroup of $G_{\mathbb{C}}$. Hence with Lemma 2.2 and Berger's list [3] we obtain:

Lemma 6.2. *Let $H < \mathrm{Sp}(p, q)$ be a subgroup of type I. Then H is symmetric and up to conjugation one of the following:*

- (1) $\mathrm{Sp}(p_1, q_1) \times \mathrm{Sp}(p_2, q_2)$ with $p_1 + p_2 = p$ and $q_1 + q_2 = q$,
- (2) $\mathrm{Sp}(p, \mathbb{C})$ if $p = q$,
- (3) $\mathrm{U}(p, q)$,
- (4) $\mathrm{GL}(p, \mathbb{H})$ for $p = q$.

6.2. Type II maximal subgroups. In this situation we have $H_{\mathbb{C}} = \mathrm{Sp}(r, \mathbb{C}) \otimes \mathrm{SO}(s, \mathbb{C})$ with $s \geq 3$, $s \neq 4$ or $(r, s) = (1, 4)$.

Lemma 6.3. *There are no real spherical subgroups $H < \mathrm{Sp}(p, q)$ of type II.*

Proof. We first claim that the real forms $H_1 = \mathrm{Sp}(r, \mathbb{R}) \otimes \mathrm{SO}(p_2, q_2)$ with $p_2 + q_2 = s$ and $H_2 = \mathrm{Sp}(r, \mathbb{R}) \otimes \mathrm{SO}^*(2m)$ with $2m = s$ are not possible. Note that $\mathrm{Sp}(r, \mathbb{R}) \subset \mathrm{SU}(r, r)$, $\mathrm{SO}(p_2, q_2) \subset \mathrm{SU}(p_2, q_2)$, $\mathrm{SO}^*(2m) \subset \mathrm{SU}(m, m)$. It follows that both H_1 and H_2 leave a Hermitian form on $\mathbb{C}^n = \mathbb{C}^{2r} \otimes \mathbb{C}^s$ invariant which is of type (rs, rs) and hence $p = q = \frac{1}{2}rs$. The dimension bound then gives

$$2r^2 + r + \frac{1}{2}s(s-1) \geq 4p^2 - p$$

with $rs = 2p$. This has no solutions since for $r \geq 1$, $s \geq 3$ we find

$$2r^2 + r + \frac{1}{2}s^2 \leq \frac{2}{9}r^2s^2 + \frac{1}{9}r^2s^2 + \frac{1}{2}r^2s^2 = \frac{5}{6}r^2s^2 \leq r^2s^2 - \frac{1}{2}rs$$

and thus neither H_1 nor H_2 can be spherical.

The case where $H = \mathrm{Sp}(p_1, q_1) \otimes \mathrm{SO}^*(2m)$ is similar, as H leaves a Hermitian form of equal parity invariant.

This leaves us with the last case where $H = \mathrm{Sp}(p_1, q_1) \otimes \mathrm{SO}(p_2, q_2)$. It requires a more detailed investigation. We request $p_1 \leq q_1$ and $q_2 \leq p_2$. Then $p_1p_2 + q_1q_2 \leq p_1q_2 + p_2q_1$ and

$$(2p_1p_2 + 2q_1q_2, 2p_1q_2 + 2p_2q_1) = (2p, 2q)$$

as H leaves invariant a Hermitian form of this signature. Inserting that in the dimension bound gives

$$2r^2 + r + \frac{1}{2}s(s-1) \geq 4p_2q_2(p_1^2 + q_1^2) + 4p_1q_1(p_2^2 + q_2^2) - (p_1p_2 + q_1q_2).$$

As in (4.3) we deduce that one factor must be compact. Suppose first that $q_2 = 0$ hence $H = \mathrm{Sp}(p_1, q_1) \otimes \mathrm{SO}(s)$ with $s = p_2$. The dimension bound in this case is:

$$2r^2 + r + \frac{1}{2}s(s-1) \geq 4q_1p_1s^2 - p_1s.$$

There are no solutions for $2r \leq s$. For $s \leq 2r$, a matrix computation (use an analogue of the map (4.5)) combined with Lemma 6.1 yields that $\mathrm{SO}(s)$ needs to have an orbit of real co-dimension at most 5 on $\mathrm{Herm}(s, \mathbb{C})$. The orbits of maximal dimension are in $\mathrm{Sym}(s, \mathbb{R})$, and they have codimension s in this space, hence $\frac{1}{2}s(s+1)$ in $\mathrm{Herm}(s, \mathbb{C})$. It follows that no $s \geq 3$ meets the requirement.

Finally we investigate the case where $p_1 = 0$. Then $r = q_1$ and $H = \mathrm{Sp}(r) \otimes \mathrm{SO}(p_2, q_2)$ and the dimension inequality becomes:

$$2r^2 + r + \frac{1}{2}s(s-1) \geq 4q_2p_2r^2 - p_2r$$

There is no solution if $3 \leq s \leq 2r$ so we may assume that $s > 2r$. With the matrix computations similar to the $\mathrm{SU}(p, q)$ -case (see (4.5)) combined with Lemma 6.1 this reduces matters to study $\mathrm{Sp}(r)$ -orbits on $\mathrm{Herm}(2r, \mathbb{C})$ with co-dimension at most 5. This implies $r = 1$, i.e. $H = \mathrm{Sp}(1) \otimes \mathrm{SO}(p, q)$ with $p+q \geq 3$. We now proceed as in Lemma 4.6: Consider $H' := \mathbf{1} \otimes \mathrm{SO}(p, q) < H$. Then H' is required to have an orbit on G/P of co-dimension at most 3. We now produce many H' -invariant functions on G/P . First we decompose $V = \mathbb{C}^n = \mathbb{C}^{p+q} \oplus \mathbb{C}^{p+q}$ into H' -orthogonal summands and write $p_i : V \rightarrow \mathbb{C}^{p+q}$, $1 \leq i \leq 2$ for the two H' -equivariant projections. Now given a flag $\mathcal{F} = \{V_1 \subset \dots \subset V_p\}$, we choose an orthonormal basis v_1, \dots, v_{2p} of V_p with respect to the standard Hermitian inner product on \mathbb{C}^n such that V_i is spanned by v_1, \dots, v_{2i} . Denote by $(\cdot, \cdot)_{p,q}$ the complex bilinear form on \mathbb{C}^{p+q} which is invariant for H' .

Then for $1 \leq m \leq p$ and $1 \leq j, k \leq 2$ we consider the function

$$g_{mjk}(\mathcal{F}) := (p_j(v_1 \wedge \dots \wedge v_{2m}), p_k(v_1 \wedge \dots \wedge v_{2m}))_{p,q}.$$

Similarly as before the rational functions

$$f_{mjk} := \mathrm{Re} \left(\frac{g_{mjk}}{g_{m11}} \right) \quad \text{and} \quad f'_{mjk} := \mathrm{Im} \left(\frac{g_{mjk}}{g_{m11}} \right)$$

are all H' -invariant. Already for $m = 1$ we obtain 4 independent invariants this way and thus H' cannot have an orbit of co-dimension 3. \square

6.3. Type III maximal subgroups.

Lemma 6.4. *Let $G = \mathrm{Sp}(p, q)$. Then there exist no real spherical subgroups of type III.*

Proof. We may assume that $1 < p$ as it is known for $p = 1$ by [17]. Then $2 \leq p \leq q$ implies $3p + 2q \leq 5q < 8q \leq 4pq$ and hence

$$4pq - p > \dim V = 2p + 2q.$$

Hence, we get from (6.1) the strict inequality $\dim H_{\mathbb{C}} > \dim V$ and again, we can use Proposition 3.3 and Remark 3.5. We are thus left with testing some sporadic cases and it is easy to see that they never satisfy the dimension bound. \square

This concludes the proof of Theorem 1.3 for $G_{\mathbb{C}} = \mathrm{Sp}(n, \mathbb{C})$.

7. THE MAXIMAL REAL SPHERICAL SUBALGEBRAS OF THE EXCEPTIONAL LIE ALGEBRAS

Here \mathfrak{g} is such that $\mathfrak{g}_{\mathbb{C}}$ is exceptional simple. We assume that \mathfrak{g} is not compact.

Lemma 7.1. *Let \mathfrak{g} be a non-complex exceptional non-compact simple real Lie algebra and \mathfrak{h} a real spherical subalgebra which is maximal reductive subalgebra. Then,*

- (1) *If $\mathfrak{g} \neq G_2^1$, then \mathfrak{h} is symmetric.*
- (2) *If $\mathfrak{g} = G_2^1$, then \mathfrak{h} is symmetric or conjugate to either $\mathfrak{h}_1 = \mathfrak{su}(2, 1)$ or $\mathfrak{h}_2 = \mathfrak{sl}(3, \mathbb{R})$ which are both absolutely spherical but not symmetric.*

Proof. If \mathfrak{g} is quasi-split, then this follows from Lemma 2.13 combined with the work of Krämer [16] (see Table 6). In particular G_2^1 , the only non-compact real form of $G_2^{\mathbb{C}}$, is split, and thus the assertion (2) is obtained with Table 7.

From now on we assume that \mathfrak{g} is not quasi-split. For $\mathfrak{g}_{\mathbb{C}} = F_4^{\mathbb{C}}$ the only non-split real form F_4^2 has real rank one, and for that the result is given in [17], Lemma 6.2. This leaves us to consider for \mathfrak{g} only the real forms E_6^3, E_6^4 of $E_6^{\mathbb{C}}$, E_7^2, E_7^3 of $E_7^{\mathbb{C}}$ and E_8^2 of $E_8^{\mathbb{C}}$.

We follow [18]. According to Dynkin [8], a subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$ is called *regular*, if it is normalized by a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. On the other hand, $\mathfrak{h}_{\mathbb{C}}$ is called an *S-subalgebra* of $\mathfrak{g}_{\mathbb{C}}$ if it is not contained in any proper regular subalgebra of $\mathfrak{g}_{\mathbb{C}}$.

Let $\mathfrak{h}_{\mathbb{C}}$ be a maximal reductive subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Then it is either regular or an S-subalgebra. According to [8] Theorem 14.1, the pairs $(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$, where $\mathfrak{h}_{\mathbb{C}}$ is non-symmetric and a maximal S-subalgebra of $E_6^{\mathbb{C}}, E_7^{\mathbb{C}}$ or $E_8^{\mathbb{C}}$, are given by:

$$\begin{aligned} (E_6^{\mathbb{C}} : A_1, G_2^{\mathbb{C}}, A_2 \oplus G_2^{\mathbb{C}}), \\ (E_7^{\mathbb{C}} : A_1, A_1 \oplus A_1, A_2, G_2^{\mathbb{C}} \oplus C_3, A_1 \oplus F_4^{\mathbb{C}}, A_1 \oplus G_2^{\mathbb{C}}), \\ (E_8^{\mathbb{C}} : A_1, A_1 \oplus A_2, B_2, G_2^{\mathbb{C}} \oplus F_4^{\mathbb{C}}), \end{aligned}$$

and by [8] Theorem 5.5 (together with the correction on p. 311 of the selected works) the pairs $(\mathfrak{g}_{\mathbb{C}} : \mathfrak{h}_{\mathbb{C}})$, where $\mathfrak{h}_{\mathbb{C}}$ is non-symmetric, semisimple, and a maximal regular subalgebra, are given by:

$$\begin{aligned} (E_6^{\mathbb{C}} : A_2 \oplus A_2 \oplus A_2), \\ (E_7^{\mathbb{C}} : A_2 \oplus A_5), \\ (E_8^{\mathbb{C}} : A_8, A_4 \oplus A_4, A_2 \oplus E_6^{\mathbb{C}}). \end{aligned}$$

Note that a maximal reductive subalgebra of $\mathfrak{g}_{\mathbb{C}}$ is either a semisimple maximal subalgebra or a maximal Levi subalgebra of $\mathfrak{g}_{\mathbb{C}}$. From the Dynkin diagram of \mathfrak{g} we can read the maximal Levi subalgebras and deduce that they are either symmetric or are contained in a semisimple maximal regular subalgebra listed above (see [4], Table on p. 219). Hence the two lists together consist in fact of all maximal reductive subalgebras which are not symmetric.

Next we record the dimension bounds obtained from (2.1):

$$\begin{aligned} (7.1) \quad \dim H &\geq 30 & (\mathfrak{g} = E_6^3) \\ (7.2) \quad \dim H &\geq 24 & (\mathfrak{g} = E_6^4) \\ (7.3) \quad \dim H &\geq 60 & (\mathfrak{g} = E_7^2) \\ (7.4) \quad \dim H &\geq 51 & (\mathfrak{g} = E_7^3) \\ (7.5) \quad \dim H &\geq 108 & (\mathfrak{g} = E_8^2) \end{aligned}$$

Going through the lists of maximal regular- and S-subalgebras of $\mathfrak{g}_{\mathbb{C}}$, we see that only the following two pairs $(G, H_{\mathbb{C}})$ satisfy the bound and thus may correspond to real spherical pairs:

$$(\mathbf{E}_6^4, \mathbf{A}_2 \oplus \mathbf{A}_2 \oplus \mathbf{A}_2) \quad \text{and} \quad (\mathbf{E}_7^3, \mathbf{A}_1 \oplus \mathbf{F}_4^{\mathbb{C}}).$$

We claim that $G = \mathbf{E}_6^4$ and a real form in G of $H_{\mathbb{C}} = \mathrm{SL}(3, \mathbb{C}) \times \mathrm{SL}(3, \mathbb{C}) \times \mathrm{SL}(3, \mathbb{C})$ cannot correspond to a real spherical pair. By inspecting the Satake diagram of \mathbf{E}_6^4 we see that the minimal parabolic P of G is contained in the maximal parabolic $P_{\max, \mathbb{C}}$ of $G_{\mathbb{C}}$, which is related to the 27-dimensional fundamental representation of $G_{\mathbb{C}}$. This representation is prehomogeneous, see case (22) in Table 3 of Proposition 3.3. If the pair was spherical then \mathbb{C}^{27} would thus become a prehomogeneous vector space for $H_{\mathbb{C}}$. As $\dim H_{\mathbb{C}} = 24 < 26$ this is excluded.

This leaves us with the case $(\mathbf{E}_7^3, \mathbf{A}_1 \oplus \mathbf{F}_4^{\mathbb{C}})$. An inspection of the Satake diagram of $G = \mathbf{E}_7^3$ shows that the minimal parabolic P of G is contained in the maximal parabolic $P_{\max, \mathbb{C}}$ of $G_{\mathbb{C}}$, which is related to the 56-dimensional fundamental representation of $G_{\mathbb{C}}$. Again this is prehomogeneous, see case (23) in Table 3. If the pair was spherical then \mathbb{C}^{56} would thus be a prehomogeneous vector space for $H_{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C}) \times \mathbf{F}_4^{\mathbb{C}}$. In particular, every irreducible $H_{\mathbb{C}}$ -submodule of \mathbb{C}^{56} is then prehomogeneous. We recall from [20], that $\mathbf{F}_4^{\mathbb{C}}$ does not admit a prehomogeneous vector space in the generalized sense: for no non-trivial irreducible representation V of $\mathbf{F}_4^{\mathbb{C}}$ and for no $n \in \mathbb{N}$ does $\mathbf{F}_4^{\mathbb{C}} \times \mathrm{GL}(n, \mathbb{C})$ admit an open orbit on $V \otimes \mathbb{C}^n$. In particular, \mathbb{C}^{56} cannot be prehomogeneous for $H_{\mathbb{C}}$. \square

The lemmas in Sections 4-7 together with the list of Krämer [16] finally conclude the proofs of Theorem 1.3 and Lemma 1.4.

8. TABLES FOR $L \cap H$

We recall the adapted parabolic subgroup $Q = L \ltimes U$ of $Z = G/H$. In view of Proposition 2.9 the Lie algebras $\mathfrak{l} \cap \mathfrak{h}$ are of central importance of us. In the following Tables 4-5 (classical and exceptional) we list all symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ (from Berger's list [3]) with \mathfrak{g} not quasi-split, together with the associated subalgebra $\mathfrak{l} \cap \mathfrak{h}$.

Following are some remarks on how the intersections $\mathfrak{l} \cap \mathfrak{h}$ have been calculated. For this we made extensive use of [13], especially §10. The complexification $Z_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ of $Z = G/H$ is by assumption still spherical. Its Luna diagram is a collection of various data of which we need only two. First, a subset $S^{(p)}$ of the set S of simple roots of G whose elements are called the *parabolic roots of Z* . Second, a finite set Σ_Z of characters of G , called the *spherical roots of Z* . Each spherical root is an \mathbb{N} -linear combination of simple roots.

The real structure provides us with the set $S^0 \subseteq S$ of compact simple roots (the black dots in the Satake diagram). Then, as mentioned already in Remark 2.12, the set of simple roots of L is the union $S^0 \cup S^{(p)}$. Now let $\Sigma_Z^0 \subseteq \Sigma_Z$ be the set of those spherical roots which lie in the span of $S^0 \cup S^{(p)}$. Then [13], Cor. 10.16 implies that $Z^0 := L/L \cap H$ is an absolutely spherical variety whose Luna diagram has still $S^{(p)}$ as set of parabolic roots and Σ_Z^0 as set of spherical roots. Since $L \cap H$ is reductive, these two data suffice to determine the isomorphism type of the derived subgroup $(L \cap H)'$ by use of tables in [5].

To determine the connected center C of $L \cap H$ it suffices to know its dimension and its real rank. The local structure theorem implies that $L/L \cap H$ is an open subset of the double coset space $U \backslash G/H$ where U is the unipotent radical of the adapted parabolic of Z . From

this we get the

$$\dim L \cap H = \dim H - \dim U = \dim H - \frac{1}{2}(\dim G - \dim L).$$

Knowing $(L \cap H)'$ we get $\dim C$. For its real rank, we use

$$\text{rank}_{\mathbb{R}} C = \text{rank}_{\mathbb{R}} L - \text{rank}_{\mathbb{R}} Z = \text{rank}_{\mathbb{R}} L - \dim \langle \text{res}_A \sigma \mid \sigma \in \Sigma_Z \rangle_{\mathbb{Q}}.$$

(see [13], Lemma 4.18). Here $A \subseteq L$ is a maximally split subtorus.

	\mathfrak{g}	\mathfrak{h}	$\mathfrak{l} \cap \mathfrak{h}$	
(1)	$\mathfrak{su}(p, q)$	$\mathfrak{so}(p, q)$	$\mathfrak{so}(q - p)$	$1 \leq p \leq q - 2$
(2)	$\mathfrak{sl}(n, \mathbb{H})$	$\mathfrak{so}^*(2n)$	$\mathfrak{u}(1)^n$	$n \geq 3$
(3)	$\mathfrak{su}(2p, 2q)$	$\mathfrak{sp}(p, q)$	$\mathfrak{sp}(q - p) + \mathfrak{sl}(2, \mathbb{C})^p$	$1 \leq p \leq q - 1$
(4)	$\mathfrak{sl}(n, \mathbb{H})$	$\mathfrak{sp}(n - k, k)$	$\mathfrak{sp}(1)^n$	$0 \leq k \leq \frac{n}{2}$
(5)	$\mathfrak{su}(p, q)$	$\mathfrak{s}[\mathfrak{u}(p_1, q_1) + \mathfrak{u}(p_2, q_2)]$	$\begin{cases} \mathfrak{s}[\mathfrak{u}(p_2 - q_1, q_2 - p_1) + \mathfrak{u}(1)^{p_1+q_1}] \\ \mathfrak{s}[\mathfrak{u}(q_1 - p_2) + \mathfrak{u}(q_2 - p_1) + \mathfrak{u}(1)^{p_1+p_2}] \end{cases}$	$\begin{cases} p_2 \geq q_1 \\ p_2 \leq q_1 \end{cases}$
(6a)	$\mathfrak{sl}(n, \mathbb{H})$	$\mathfrak{s}[\mathfrak{gl}(n - k, \mathbb{H}) + \mathfrak{gl}(k, \mathbb{H})]$	$\mathfrak{s}[\mathfrak{gl}(n - 2k, \mathbb{H}) + \mathfrak{gl}(1, \mathbb{H})^k]$	$0 \leq k \leq \frac{n}{2}$
(6b)	$\mathfrak{sl}(n, \mathbb{H})$	$\mathfrak{sl}(n, \mathbb{C}) + \mathfrak{u}(1)$	$\begin{cases} \mathfrak{s}[\mathfrak{gl}(1, \mathbb{H})^{\frac{n}{2}}] \\ \mathfrak{s}[\mathfrak{gl}(1, \mathbb{H})^{\lfloor \frac{n}{2} \rfloor}] + \mathfrak{u}(1) \end{cases}$	$\begin{cases} n \text{ even} \\ n \text{ odd} \end{cases}$
(7)	$\mathfrak{so}(2p, 2q)$	$\mathfrak{u}(p, q)$	$\mathfrak{u}(q - p) + \mathfrak{su}(1, 1)^p$	$1 \leq p \leq q - 2$
(8a)	$\mathfrak{so}^*(2n)$	$\mathfrak{u}(n - k, k)$	$\begin{cases} \mathfrak{su}(2)^{\frac{n}{2}}, n \text{ and } k \text{ even} \\ \mathfrak{su}(2)^{\frac{n}{2}-1} + \mathfrak{su}(1, 1) + \mathfrak{u}(1), n \text{ even}, k \text{ odd} \\ \mathfrak{su}(2)^{\frac{n-1}{2}} + \mathfrak{u}(1), n \text{ odd} \end{cases}$	$0 \leq k \leq \frac{n}{2}, n \geq 4$
(8b)	$\mathfrak{so}^*(2n)$	$\mathfrak{gl}(\frac{n}{2}, \mathbb{H})$	$\mathfrak{su}(2)^{\frac{n}{2}}$	$n \geq 4 \text{ even}$
(9)	$\mathfrak{so}(p, q)$	$\mathfrak{so}(p_1, q_1) + \mathfrak{so}(p_2, q_2)$	$\begin{cases} \mathfrak{so}(p_2 - q_1, q_2 - p_1) \\ \mathfrak{so}(q_1 - p_2) + \mathfrak{so}(q_2 - p_1) \end{cases}$	$\begin{cases} p_2 \geq q_1 \\ p_2 \leq q_1 \end{cases}$
(10a)	$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2n - 2k) + \mathfrak{so}^*(2k)$	$\mathfrak{so}^*(2n - 4k) + \mathfrak{u}(1)^k$	$1 \leq k \leq \frac{n}{2}$
(10b)	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n, \mathbb{C})$	$\mathfrak{u}(1)^{\lfloor \frac{n}{2} \rfloor}$	$n \geq 4$
(11)	$\mathfrak{sp}(p, q)$	$\begin{cases} \mathfrak{u}(p, q) \\ \mathfrak{gl}(p, \mathbb{H}), p = q \end{cases}$	$\mathfrak{u}(q - p) + \mathfrak{u}(1)^p$	$1 \leq p \leq q$
(12a)	$\mathfrak{sp}(p, q)$	$\mathfrak{sp}(p_1, q_1) + \mathfrak{sp}(p_2, q_2)$	$\begin{cases} \mathfrak{sp}(p_2 - q_1, q_2 - p_1) + \mathfrak{sp}(1)^{p_1+q_1} \\ \mathfrak{sp}(q_1 - p_2) + \mathfrak{sp}(q_2 - p_1) + \mathfrak{sp}(1)^{p_1+p_2} \end{cases}$	$\begin{cases} p_2 \geq q_1 \\ p_2 \leq q_1 \end{cases}$
(12b)	$\mathfrak{sp}(p, p)$	$\mathfrak{sp}(p, \mathbb{C})$	$\mathfrak{sp}(1)^p$	

Table 4

The notation in Table 4 follows certain conventions: in each row where the letters appear one has $p = p_1 + p_2 \leq q = q_1 + q_2$ and $p_1 + q_1 \leq p_2 + q_2$ (whence $p_1 \leq q_2$).

\mathfrak{g}	\mathfrak{h}	$\mathfrak{l} \cap \mathfrak{h}$
E_6^3	$\mathfrak{sp}(2, 2)$	$\mathfrak{so}(4)$
E_6^4	$\mathfrak{sp}(3, 1)$	$\mathfrak{so}(4) + \mathfrak{so}(4)$
E_6^3	$\begin{cases} \mathfrak{su}(4, 2) + \mathfrak{su}(2) \\ \mathfrak{su}(5, 1) + \mathfrak{sl}(2, \mathbb{R}) \end{cases}$	$\mathfrak{u}(2) + \mathfrak{u}(2)$
E_6^4	$\mathfrak{sl}(3, \mathbb{H}) + \mathfrak{su}(2)$	$\mathfrak{so}(5) + \mathfrak{so}(3) + \mathfrak{gl}(1, \mathbb{R})$
E_6^3	$\begin{cases} \mathfrak{so}(10) + \mathfrak{u}(1) \\ \mathfrak{so}(8, 2) + \mathfrak{u}(1) \\ \mathfrak{so}^*(10) + \mathfrak{u}(1) \end{cases}$	$\mathfrak{u}(4)$
E_6^4	$\mathfrak{so}(9, 1) + \mathfrak{gl}(1, \mathbb{R})$	$\mathfrak{spin}(7) + \mathfrak{gl}(1, \mathbb{R})$
E_6^3	F_4^2	$\mathfrak{so}(7, 1)$
E_6^4	$\begin{cases} F_4^2 \\ F_4 \end{cases}$	$\mathfrak{so}(8)$
E_7^2	$\begin{cases} \mathfrak{su}(6, 2) \\ \mathfrak{su}(4, 4) \end{cases}$	$\mathfrak{so}(2)^3$
E_7^3	$\begin{cases} \mathfrak{su}(6, 2) \\ \mathfrak{sl}(4, \mathbb{H}) \end{cases}$	$\mathfrak{so}(4) + \mathfrak{so}(4)$
E_7^2	$\begin{cases} \mathfrak{so}(12) + \mathfrak{su}(2) \\ \mathfrak{so}(8, 4) + \mathfrak{su}(2) \\ \mathfrak{so}^*(12) + \mathfrak{sl}(2, \mathbb{R}) \end{cases}$	$\mathfrak{su}(2)^3$
E_7^3	$\begin{cases} \mathfrak{so}(10, 2) + \mathfrak{sl}(2, \mathbb{R}) \\ \mathfrak{so}^*(12) + \mathfrak{su}(2) \end{cases}$	$\mathfrak{so}(6) + \mathfrak{so}(2) + \mathfrak{sl}(2, \mathbb{R})$
E_7^2	$\begin{cases} E_6^2 + \mathfrak{u}(1) \\ E_6^3 + \mathfrak{u}(1) \end{cases}$	$\mathfrak{so}(6, 2) + \mathfrak{so}(2)$
E_7^3	$\begin{cases} E_6^3 + \mathfrak{u}(1) \\ E_6^4 + \mathfrak{gl}(1, \mathbb{R}) \\ E_6 + \mathfrak{u}(1) \end{cases}$	$\mathfrak{so}(8)$
E_8^2	$\begin{cases} \mathfrak{so}(12, 4) \\ \mathfrak{so}^*(16) \end{cases}$	$\mathfrak{so}(4) + \mathfrak{so}(4)$
E_8	$\mathfrak{so}(16)$	$\mathfrak{so}(16)$
E_8^2	$\begin{cases} E_7^2 + \mathfrak{su}(2) \\ E_7^3 + \mathfrak{sl}(2, \mathbb{R}) \\ E_7 + \mathfrak{su}(2) \end{cases}$	$\mathfrak{so}(8)$
F_4^2	$\mathfrak{sp}(2, 1) + \mathfrak{su}(2)$	$\mathfrak{so}(4) + \mathfrak{so}(3)$
F_4^2	$\begin{cases} \mathfrak{so}(9) \\ \mathfrak{so}(8, 1) \end{cases}$	$\mathfrak{spin}(7)$

Table 5

In most cases, it is not necessary to know the embedding of $\mathfrak{l} \cap \mathfrak{h}$ into \mathfrak{h} but in some it does matter. For this, the following lemma is useful.

Lemma 8.1. *Let $\mathfrak{h}_1, \mathfrak{h}_2$ be two self-normalizing real spherical subalgebras of \mathfrak{g} with adapted parabolic subalgebras \mathfrak{q}_1 and \mathfrak{q}_2 . Suppose that $\mathfrak{h}_{1,\mathbb{C}} \in \text{Ad}(G_{\mathbb{C}})\mathfrak{h}_{2,\mathbb{C}}$. Then there exists an element $g \in G_{\mathbb{C}}$ of the form $g = tg_0$ with $g_0 \in G$ and $t \in Z(L_{2,\mathbb{C}})$ such that $\text{Ad}(g)$ maps $\mathfrak{h}_{1,\mathbb{C}}, \mathfrak{q}_{1,\mathbb{C}}, \mathfrak{l}_1 \cap \mathfrak{h}_1$ onto $\mathfrak{h}_{2,\mathbb{C}}, \mathfrak{q}_{2,\mathbb{C}}, \mathfrak{l}_2 \cap \mathfrak{h}_2$.*

Proof. [13], Section 13. □

In particular, the lemma says that the complexification of the embedding $\mathfrak{l} \cap \mathfrak{h} \hookrightarrow \mathfrak{h}$ does not depend on the particular real form \mathfrak{h} . This is used in part (a) of the following remark.

Remark 8.2. (a) In Table 5 there is ambiguity how $\mathfrak{l} \cap \mathfrak{h}$ is embedded into \mathfrak{h} in some cases where $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ consists of two factors. However, with Lemma 8.1 one can derive the following additional data from the table:

- (a₁) For $\mathfrak{g} = E_6^3$ and $\mathfrak{h}_{\mathbb{C}} = \mathfrak{sl}(6, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ one has $[\mathfrak{l} \cap \mathfrak{h}, \mathfrak{l} \cap \mathfrak{h}] \subset \mathfrak{h}_1$.
- (a₂) For $\mathfrak{g} = E_7^2$ and $\mathfrak{h}_{\mathbb{C}} = \mathfrak{so}(12, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ one has $\mathfrak{l} \cap \mathfrak{h} \subset \mathfrak{h}_1$.
- (a₃) For $\mathfrak{g} = E_7^3$ and $\mathfrak{h}_{\mathbb{C}} = \mathfrak{so}(12, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, one has $[\mathfrak{l} \cap \mathfrak{h}, \mathfrak{l} \cap \mathfrak{h}] \subset \mathfrak{h}_1$.

To see that, we discuss the case (a₁). The arguments for (a₂) and (a₃) are similar. Table 5 shows that there are two symmetric subalgebras in \mathfrak{g} , say $\mathfrak{h}' = \mathfrak{h}'_1 \oplus \mathfrak{h}'_2 = \mathfrak{su}(4, 2) \oplus \mathfrak{su}(2)$ and $\mathfrak{h}'' = \mathfrak{h}''_1 \oplus \mathfrak{h}''_2 = \mathfrak{su}(5, 1) \oplus \mathfrak{sl}(2, \mathbb{R})$ with isomorphic complexifications. By Lemma 8.1 there is an isomorphism of $\mathfrak{g}_{\mathbb{C}}$ which carries $\mathfrak{h}''_{\mathbb{C}}$ to $\mathfrak{h}'_{\mathbb{C}}$ and $\mathfrak{l}'' \cap \mathfrak{h}''$ to $\mathfrak{l}' \cap \mathfrak{h}'$. Table 5 shows that $\mathfrak{l} \cap \mathfrak{h}$ is of compact type, and hence $[\mathfrak{l}'' \cap \mathfrak{h}'', \mathfrak{l}'' \cap \mathfrak{h}''] \subset \mathfrak{h}''_1$ by Schur's lemma. It then follows that $[\mathfrak{l}' \cap \mathfrak{h}', \mathfrak{l}' \cap \mathfrak{h}'] \subset \mathfrak{h}'_1$.

(b) For $\mathfrak{g} = F_4^2$ and $\mathfrak{h} = \mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2)$ the algebra $\mathfrak{l} \cap \mathfrak{h}$ surjects onto \mathfrak{h}_2 . To see this let V be the 52-dimensional irreducible representation of F_4 . Then a computation with LiE using the inclusions $\mathfrak{l} \cap \mathfrak{h} \subseteq \mathfrak{l} \subseteq \mathfrak{g}$ shows that $\dim V^{\mathfrak{l} \cap \mathfrak{h}} = 1$. On the other hand, $\text{res}_{\mathfrak{h}}^{\mathfrak{g}} V$ contains the 3-dimensional $\mathfrak{sp}(3) \oplus \mathfrak{sl}(2)$ -module $\mathbb{C} \otimes S^2 \mathbb{C}^2$. This cannot happen if the projection of $\mathfrak{l} \cap \mathfrak{h}$ to \mathfrak{h}_2 were trivial.

(c) In the classical case (Table 4) there are also some situations where \mathfrak{h} is not simple, and where it is of interest how certain factors of $\mathfrak{l} \cap \mathfrak{h}$ are embedded into \mathfrak{h} . These are:

- In (5) $\mathfrak{u}(1)^{p_1+q_1}$, resp. $\mathfrak{u}(1)^{p_1+p_2}$, is diagonally embedded into $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$,
- In (6a) $\mathfrak{gl}(1, \mathbb{H})^k$ is diagonally embedded into $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$,
- In (10a) $\mathfrak{u}(1)^k$ is diagonally embedded into $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$,
- In (12a) $\mathfrak{sp}(1)^{p_1+q_1}$, resp. $\mathfrak{sp}(1)^{p_1+p_2}$, is diagonally embedded into $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$.

This can be verified as follows: Let σ be the involution which determines \mathfrak{h} and let θ be the standard Cartan involution which commutes with σ . Let \mathfrak{k} be the fixed point set of θ . Then \mathfrak{l} can be chosen as the centralizer of a generic element $X \in \mathfrak{h}^{\perp} \cap \mathfrak{k}^{\perp}$. Simple matrix computations then verify the bulleted assertions.

(d) In the last two lines of Table 5 we have $\mathfrak{l} \cap \mathfrak{h} = \mathfrak{spin}(7)$. That it is the spin embedding (and not $\mathfrak{so}(7)$) is seen in both cases from the fact that the complement in \mathfrak{g} contains the spin representation.

9. THE CLASSIFICATION OF REDUCTIVE REAL SPHERICAL SUBGROUPS

Now, after we have classified all maximal spherical subalgebras which are reductive we can complete the classification. This section is divided into three parts: classical, exceptional and complex.

We recall the adapted parabolic $Q = L \ltimes U \supset P$ of a real spherical space. We set $L_H := L \cap H$ and denote its Lie algebra by \mathfrak{l}_h . Further we may assume that $MA \subset L$.

The general strategy is as follows. Given G and a maximal reductive real spherical subgroup H we let $H' < H$ be a proper reductive subgroup. According to Proposition 2.9 the space G/H' is real spherical if and only if H/H' is a real spherical L_H -variety. In particular,

$$(9.1) \quad \mathfrak{h} = \mathfrak{h}' + \mathfrak{l}_h$$

needs to hold by Corollary 2.10. By Lemma 1.4, H is symmetric in almost all cases, and hence \mathfrak{l}_h is given by the tables in Section 8. By Proposition 2.5 we can then determine whether (9.1) is valid and thus limit the number of subgroups H' to consider.

In addition to (9.1) there is a second general fact which will be useful in the classification. Let us call \mathfrak{h} *almost absolutely spherical* if it is real spherical and there exists an absolutely spherical subalgebra $\bar{\mathfrak{h}}$ of \mathfrak{g} with $[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subseteq \mathfrak{h} \subseteq \bar{\mathfrak{h}}$.

Lemma 9.1. *Let \mathfrak{g} be a non-compact and non-complex simple Lie algebra and \mathfrak{h} a reductive subalgebra which is not absolutely spherical. Then $(\mathfrak{g}, \mathfrak{h})$ is almost absolutely spherical if and only if it is isomorphic to one of the pairs in Table 1 of Theorem 1.1 which are marked by an asterisk.*

Proof. We use the real version of the Vinberg-Kimel'feld criterion (see [14] Prop. 3.7): the subalgebra \mathfrak{h} is real spherical if and only if $\dim V^{\mathfrak{h}} \leq 1$ for all P -spherical simple representations of G . Observe, that it suffices to check this condition over \mathbb{C} .

Now let $\bar{\mathfrak{h}} \subseteq \mathfrak{g}$ be an absolutely spherical subalgebra in which \mathfrak{h} is coabelian. Because \mathfrak{h} is not absolutely spherical the complexified pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ will appear (according to Krämer [16]) in the following table:

$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{h}_{\mathbb{C}}$	$\overline{\mathcal{M}}$	α
$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) + \mathfrak{sl}(n, \mathbb{C})$	$\omega_1, \dots, \omega_{n-1}, \omega_n \pm \epsilon, \omega_{n+1}, \dots, \omega_{2n-1}$	α_n
$\mathfrak{so}(4n, \mathbb{C})$	$\mathfrak{sl}(2n, \mathbb{C})$	$\omega_2, \omega_4, \dots, \omega_{2n-2}, \omega_{2n} \pm \epsilon$	α_{2n}
$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C})$	$\omega_1, \dots, \omega_{n-1}, \omega_n \pm \epsilon$	α_n
$\mathfrak{so}(n, \mathbb{C})$	$\mathfrak{so}(n-2, \mathbb{C})$	$\omega_1 \pm \epsilon, \omega_2$	α_1
$\mathfrak{so}(10, \mathbb{C})$	$\mathfrak{spin}(7, \mathbb{C})$	$\omega_1 \pm 2\epsilon, \omega_2, \omega_4 + \epsilon, \omega_5 - \epsilon$	α_1
$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C})$	$2\omega_1, \dots, 2\omega_{n-1}, \omega_n \pm \epsilon$	α_n
$\mathfrak{sp}(n+1, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{C})$	$\omega_1 \pm \epsilon, \omega_2$	α_1
$E_7^{\mathbb{C}}$	$E_6^{\mathbb{C}}$	$\omega_1, \omega_6, \omega_7 \pm \epsilon$	α_7

Since $\overline{H}_{\mathbb{C}}$ normalizes $X = G_{\mathbb{C}}/H_{\mathbb{C}}$, there is a right action of $T_0 := \overline{H}_{\mathbb{C}}/H_{\mathbb{C}} \cong \mathbb{C}^*$ on X . Moreover, because $G_{\mathbb{C}}/\overline{H}_{\mathbb{C}}$ is (absolutely) spherical, X is spherical as $\overline{G} = G_{\mathbb{C}} \times T_0$ -variety. The corresponding weights (i.e. highest weights of irreducible \overline{G} -modules V containing a nontrivial $\overline{H}_{\mathbb{C}}$ -fixed vector) are called the extended weights of X . They are characters of $B \times T_0$, where $B \subseteq P_{\mathbb{C}} \subseteq G_{\mathbb{C}}$ is a Borel subgroup, and form a monoid $\overline{\mathcal{M}}$. Now the third column shows the generators of this monoid. Here, ϵ generates the character group of T_0 . The expansion of a character is given w.r.t. the fundamental weights following the Bourbaki notation. The set of weights \mathcal{M} of X as a G -variety is obtained by dropping the T_0 -components, i.e., by setting $\epsilon = 0$. This way we get a surjective map $\pi : \overline{\mathcal{M}} \rightarrow \mathcal{M}$. Let $\overline{\mathcal{M}}_P \subseteq \overline{\mathcal{M}}$ be the submonoid of weights whose first component (i.e. its restriction to B) is a weight of $P_{\mathbb{C}}$. Then the Vinberg-Kimel'feld criterion implies that G/H is real spherical if and only if the restriction of π to $\overline{\mathcal{M}}_P$ is injective.

To decide injectivity one checks that in every case there is a unique simple root α of $\mathfrak{g}_{\mathbb{C}}$ (given in the fourth column) with the property that the restriction of π to $\overline{\mathcal{M}} \cap H_{\alpha}$ is injective where H_{α} is the hyperplane perpendicular to α . We claim that G/H is real spherical if and only if α is a compact simple root of G . Indeed, if α is compact then $\overline{\mathcal{M}}_P \subseteq H_{\alpha}$ and the restriction of π is injective. Conversely, if α is non-compact then the unique fundamental weight ω with $\langle \omega, \alpha^{\vee} \rangle = 1$ is a weight of P . Moreover, by inspection of the table one sees that there is $d \geq 1$ with $\omega \pm d\epsilon \in \overline{\mathcal{M}}$. Thus the restriction of π is not injective.

Finally, the lemma is proved by simply finding all real forms of $(\mathfrak{g}_{\mathbb{C}}, \overline{\mathfrak{h}}_{\mathbb{C}})$ according to Berger or Appendix 10 for which α is a compact root of \mathfrak{g} . For example the first item yields among others the pair $(\mathfrak{sl}(n, \mathbb{H}), \mathfrak{sl}(n, \mathbb{C}))$ which is real spherical if and only if α_n is compact for $\mathfrak{sl}(n, \mathbb{H})$, hence if and only if n is odd. \square

9.1. The classical cases.

Proposition 9.2. *Let $\mathfrak{g} = \mathfrak{su}(p, q)$, $1 \leq p \leq q$, and let \mathfrak{h} be a reductive subalgebra. Then $(\mathfrak{g}, \mathfrak{h})$ is real spherical if and only if either it is absolutely spherical or \mathfrak{h} is conjugate to one of the following:*

- (i) $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 = \mathfrak{su}(p_1, q_1) \oplus \mathfrak{su}(p_2, q_2)$ with $p_1 + p_2 = p$, $q_1 + q_2 = q$ and $p_1 + q_1 = p_2 + q_2$, but $(p_1, q_1) \neq (q_2, p_2)$, or
- (ii) $p = 1$, $q = q_1 + q_2$, q_2 even and $\mathfrak{h} = \mathfrak{su}(1, q_1) \oplus \mathfrak{sp}(q_2/2) \oplus \mathfrak{f}$ with $\mathfrak{f} \subset \mathfrak{u}(1)$.

Proof. Since we shall refer to Proposition 2.9 it will be convenient to replace the notation \mathfrak{h} in the above statement by \mathfrak{h}' , and let \mathfrak{h} instead denote a maximal reductive subalgebra with $\mathfrak{h}' \subset \mathfrak{h} \subset \mathfrak{g}$. Then \mathfrak{h} is symmetric by Lemma 1.4.

We need to consider the cases (1), (3) and (5) from Table 4. For case (1) we observe that $\mathfrak{l}_{\mathfrak{h}}$ is compact but not \mathfrak{h} . Hence by Lemma 2.4 there is no proper real spherical subalgebra \mathfrak{h}' of \mathfrak{h} . We can argue similarly for (3) as symplectic algebras do not admit factorizations by Proposition 2.5.

This leaves us with (5), i.e.

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 = \mathfrak{su}(p_1, q_1) \oplus \mathfrak{su}(p_2, q_2) \oplus \mathfrak{u}(1)$$

with $p_1 + q_1, p_2 + q_2 > 0$ and $p_1 + p_2 = p$, $q_1 + q_2 = q$. Set $r := p_1 + q_1$ and $s := p_2 + q_2$. We may assume that $r \leq s$. Note that since $p \leq q$ this implies that $q_2 - p_1 \geq |p_2 - q_1|$.

According to Table 4 we have

$$(9.2) \quad \mathfrak{l}_{\mathfrak{h}} = \mathfrak{s}[\mathfrak{u}(p_2 - q_1, q_2 - p_1) \oplus \mathfrak{u}(1)^r]$$

when $p_2 \geq q_1$, and when $p_2 \leq q_1$ we have

$$(9.3) \quad \mathfrak{l}_{\mathfrak{h}} = \mathfrak{s}[\mathfrak{u}(q_1 - p_2) \oplus \mathfrak{u}(q_2 - p_1) \oplus \mathfrak{u}(1)^p].$$

Let us first consider the case where $\mathfrak{h}' \neq [\mathfrak{h}, \mathfrak{h}]$ and start with $p_2 \geq q_1$. The embedding into \mathfrak{h} of (9.2) is such that the projection of $\mathfrak{l}_{\mathfrak{h}}$ to \mathfrak{h}_2 is injective. Hence we deduce from (9.1) and Proposition 2.5 that $r = p_1 + q_1 = 1$ and $\mathfrak{h}' = \mathfrak{sp}(\frac{p_2}{2}, \frac{q_2}{2})$ or $\mathfrak{h}' = \mathfrak{sp}(\frac{p_2}{2}, \frac{q_2}{2}) \oplus \mathfrak{u}(1)$ both of which are absolutely spherical according to Table 7.

Next we consider the case where $p_2 \leq q_1$ with $\mathfrak{l}_{\mathfrak{h}}$ given by (9.3). Note that $\mathfrak{u}(1)^p$ projects injectively to both factors \mathfrak{h}_1 and \mathfrak{h}_2 . Hence we deduce from (9.1) and Proposition 2.5 that $p = p_1 + p_2 = 1$. Without loss of generality let $p_1 = 1$ and $p_2 = 0$, i.e. $\mathfrak{g} = \mathfrak{su}(1, q)$ and $\mathfrak{h} = \mathfrak{su}(1, q_1) \oplus \mathfrak{su}(q_2) \oplus \mathfrak{u}(1)$. Proposition 2.5 forces q_2 to be even and shows that

$\mathfrak{h}' = \mathfrak{h}_1 \oplus \mathfrak{sp}(q_2/2)$ or $\mathfrak{h}' = \mathfrak{h}_1 \oplus \mathfrak{sp}(q_2/2) \oplus \mathfrak{u}(1)$, both of which are real spherical. This is case (ii).

Finally let us consider the case where $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}_1 + \mathfrak{h}_2$. According to Table 7 this is absolutely spherical provided that $r < s$. For $r = s$, case (i) follows from Lemma 9.1. \square

Proposition 9.3. *Let $\mathfrak{g} = \mathfrak{sl}(m, \mathbb{H})$ for $m \geq 3$, and let \mathfrak{h} be a reductive subalgebra. Then $(\mathfrak{g}, \mathfrak{h})$ is a real spherical pair if and only if it is absolutely spherical or, up to conjugation,*

- (1) $\mathfrak{h} = \mathfrak{sl}(m-1, \mathbb{H}) \oplus \mathfrak{f}$ with $\mathfrak{f} \subset \mathbb{C}$, or
- (2) $\mathfrak{h} = \mathfrak{sl}(m, \mathbb{C})$ with m odd.

Proof. We need to treat the cases (4) and (6) from Table 4. Now, since symplectic algebras do not admit factorizations, we are left with the two cases in (6).

We begin with $\mathfrak{h} = \mathfrak{s}(\mathfrak{gl}(m_1, \mathbb{H}) \oplus \mathfrak{gl}(m_2, \mathbb{H}))$, $m = m_1 + m_2$, $m_1 \geq m_2$. Set $\mathfrak{h}_1 = \mathfrak{sl}(m_1, \mathbb{H})$, $\mathfrak{h}_2 = \mathfrak{sl}(m_2, \mathbb{H})$ and $\mathfrak{h}_3 = \mathfrak{z}(\mathfrak{h}) = \mathfrak{gl}(1, \mathbb{R})$.

Here we have from Table 4

$$\mathfrak{l}_{\mathfrak{h}} = \mathfrak{s}[\mathfrak{gl}(m_1 - m_2, \mathbb{H}) \oplus \mathfrak{gl}(1, \mathbb{H})^{m_2}]$$

with $\mathfrak{sl}(m_1 - m_2, \mathbb{H}) < \mathfrak{h}_1$ and $\mathfrak{gl}(1, \mathbb{H})^{m_2}$ diagonally embedded. According to [16], $[\mathfrak{h}, \mathfrak{h}]$ is a real form if and only if $m_1 \neq m_2$. If $m_1 = m_2$, then $\mathfrak{l}_{\mathfrak{h}}$ does not surject to the center of \mathfrak{h} and hence $[\mathfrak{h}, \mathfrak{h}]$ is not spherical. If $m_2 > 1$, then we obtain via (9.1) and Proposition 2.5 that the only possible spherical subalgebra contained in \mathfrak{h} is $[\mathfrak{h}, \mathfrak{h}]$.

In case $m_2 = 1$, $\mathfrak{m} \cap \mathfrak{h}$ surjects onto $\mathfrak{h}_2 = \mathfrak{su}(2)$ and we obtain the cases listed in (1).

The second possibility for \mathfrak{h} is $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{sl}(m, \mathbb{C})$. For that we first note that dimension bound excludes $\mathfrak{h}' := \mathfrak{u}(1) \oplus \mathfrak{h}'_2$ with \mathfrak{h}_2 a proper reductive subalgebra of $\mathfrak{sl}(m, \mathbb{C})$ to be real spherical. The cases where $[\mathfrak{h}, \mathfrak{h}]$ are spherical are deduced from Lemma 9.1. \square

Proposition 9.4. *Let $\mathfrak{g} = \mathfrak{so}^*(2m)$ for $m \geq 5$. Then a reductive subalgebra is real spherical if and only if it is absolutely spherical or conjugate to one of the following:*

- (1) $\mathfrak{h} = \mathfrak{so}^*(2m-2)$, or
- (2) $m = 5$, $\mathfrak{h} = \mathfrak{spin}(5, 2)$ or $\mathfrak{spin}(6, 1)$.

Proof. We need to consider the cases (8) and (10) from Table 4. In case of (8) there are no proper real spherical subalgebras of \mathfrak{g} contained in \mathfrak{h} by (9.1) and Proposition 2.5. In case (10) with $\mathfrak{h} = \mathfrak{so}(m, \mathbb{C})$ the dimension bound excludes a proper reductive subalgebra of \mathfrak{h} to be real spherical.

Finally we need to treat the case where $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 = \mathfrak{so}^*(2m_1) \oplus \mathfrak{so}^*(2m_2)$ with $m_1 \leq m_2$, $m_1 + m_2 = m$. Here $\mathfrak{l}_{\mathfrak{h}} = \mathfrak{so}^*(2(m_2 - m_1)) \oplus \mathfrak{so}(2)^{m_1}$ with $\mathfrak{so}^*(2(m_2 - m_1)) < \mathfrak{so}^*(2m_2)$. When $m_1 > 1$ we deduce from Propositions 2.9 and 2.5 that there exist no proper reductive subalgebras of \mathfrak{h} which are real spherical. If $m_1 = 1$, then $\mathfrak{h} = \mathfrak{so}(2, \mathbb{R}) \oplus \mathfrak{so}^*(2m_2)$, $\mathfrak{m} \cap \mathfrak{h}$ surjects onto $\mathfrak{so}(2, \mathbb{R})$ and thus $\mathfrak{so}^*(2m_2)$ is real spherical. Further factorizations are only possible for $m_2 = 4$ which results in the real spherical subalgebras $\mathfrak{spin}(6, 1)$ and $\mathfrak{spin}(5, 2)$ in $\mathfrak{h}_2 = \mathfrak{so}^*(8) \simeq \mathfrak{so}(6, 2)$. \square

Proposition 9.5. *Let $\mathfrak{g} = \mathfrak{so}(p, q)$, $1 \leq p \leq q$, $p+q \geq 6$. Then a reductive subalgebra is real spherical if and only if it is either absolutely spherical or conjugate to one of the following:*

- (1) p, q , and $\frac{p+q}{2}$ all even, $p \neq q$, and $\mathfrak{h} = \mathfrak{su}(\frac{p}{2}, \frac{q}{2})$.
- (2) $p = 2r$, $q = 2s + 1$, $r \neq s$, and $\mathfrak{h} = \mathfrak{su}(r, s)$.
- (3) $p = 2r + 1$, $q = 2s$, $r \neq s$, and $\mathfrak{h} = \mathfrak{su}(r, s)$.

- (4) $p = 1, q = q_1 + 4$ and $\mathfrak{h} = \mathfrak{so}(1, q_1) + \mathfrak{h}'$ with $\mathfrak{h}' \subsetneq \mathfrak{so}(4) \simeq \mathfrak{so}(3) \times \mathfrak{so}(3)$ a subalgebra such that $\mathfrak{h}' + \text{diag } \mathfrak{so}(3) = \mathfrak{so}(4)$.
- (5) $p = 1, q = q_1 + q_2, q_2 \geq 5$, and $\mathfrak{h} = \mathfrak{so}(1, q_1) \oplus \mathfrak{h}_2$ with $\mathfrak{h}_2 \subsetneq \mathfrak{so}(q_2)$ a subalgebra such that $\mathfrak{h}_2 + \mathfrak{so}(q_2 - 1) = \mathfrak{so}(q_2)$ (see Proposition 2.5).
- (6) $p = 2, q = q_1 + 7$ and $\mathfrak{h} = \mathfrak{so}(2, q_1) \oplus \mathbf{G}_2$.
- (7) $p = 2, q = q_1 + 8$ and $\mathfrak{h} = \mathfrak{so}(2, q_1) \oplus \mathfrak{spin}(7)$.
- (8) $p = 3, q = q_1 + 8$ and $\mathfrak{h} = \mathfrak{so}(3, q_1) \oplus \mathfrak{spin}(7)$.
- (9) $p = 3, q = 6$ and $\mathfrak{h} = \mathfrak{so}(2) \oplus \mathbf{G}_2^1$.
- (10) $p = 4, q = 7$ and $\mathfrak{h} = \mathfrak{so}(3) \oplus \mathfrak{spin}(3, 4)$.

Proof. According to Lemma 5.6 and Lemma 5.7 there are no maximal spherical subalgebras of type II or III unless \mathfrak{g} is split. Moreover, by Lemma 5.4 all maximal subalgebras of type I are symmetric. As we may assume that \mathfrak{g} is not quasisplit, we need only to consider subalgebras of the symmetric subalgebras \mathfrak{h} from cases (7) and (9) in Table 4.

We begin with the first of these, i.e. $\mathfrak{h} = \mathfrak{u}(\frac{p}{2}, \frac{q}{2})$ with $p \neq q$ even and $\mathfrak{l}_{\mathfrak{h}} = \mathfrak{u}(\frac{q-p}{2}) \oplus \mathfrak{sl}(2, \mathbb{R})^{\frac{q}{2}}$ where $\mathfrak{u}(\frac{q-p}{2}) < \mathfrak{so}(q-p) = \mathfrak{m}$. In particular, $\mathfrak{u}(\frac{q-p}{2})$ surjects onto the center of \mathfrak{h} and we deduce that $[\mathfrak{h}, \mathfrak{h}]$ is spherical. According to Krämer this is absolutely spherical if and only if $\frac{p+q}{2}$ is odd, and we obtain (1). From the structure of $\mathfrak{l}_{\mathfrak{h}}$ we deduce from Proposition 2.5 and Proposition 2.9 that no other reductive proper subalgebra $\mathfrak{h}' < \mathfrak{h}$ is real spherical in \mathfrak{g} .

This leaves us with the case (9) from Table 4, where $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 = \mathfrak{so}(p_1, q_1) \oplus \mathfrak{so}(p_2, q_2)$ with $p = p_1 + p_2 \leq q = q_1 + q_2$, $r = p_1 + q_1 \leq s = p_2 + q_2$, and $p_1 \leq q_2$. In case $p_2 \leq q_1$ we have $P = Q$ and

$$(9.4) \quad \mathfrak{l}_{\mathfrak{h}} = \mathfrak{so}(q_1 - p_2) \oplus \mathfrak{so}(q_2 - p_1).$$

In case $p_2 \geq q_1$ we have

$$(9.5) \quad \mathfrak{l}_{\mathfrak{h}} = \mathfrak{so}(p_2 - q_1, q_2 - p_1) < \mathfrak{h}_2.$$

To start with we exclude the diagonal case where $\mathfrak{h}_1 \simeq \mathfrak{h}_2$ and $\mathfrak{h}' \simeq \mathfrak{h}_1$ is "diagonally" embedded into $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. For that we note that $\mathfrak{h}_1 \simeq \mathfrak{h}_2$ either means that $\mathfrak{h}_1 = \mathfrak{h}_2$ or $(p_1, q_1) = (q_2, p_2)$. In the latter case \mathfrak{g} is split and we remain with $\mathfrak{h}_1 = \mathfrak{h}_2$. In particular \mathfrak{h} is non-compact and semisimple, but $\mathfrak{l}_{\mathfrak{h}} = \mathfrak{so}(q_1 - p_1) \oplus \mathfrak{so}(q_1 - p_1)$ is compact. Thus $H = H' L_H$ is not possible by Lemma 2.4.

We now begin with the case of $p_2 \leq q_1$ and (9.4). Suppose that $\mathfrak{h}_1 \oplus \mathfrak{h}'_2$ is a spherical subalgebra of \mathfrak{g} , with $\mathfrak{h}'_2 \subsetneq \mathfrak{h}_2$. Then $\mathfrak{h}_2 = \mathfrak{h}'_2 + \mathfrak{l}_{\mathfrak{h}}^2$ with $\mathfrak{l}_{\mathfrak{h}}^2 = \mathfrak{so}(q_2 - p_1)$, i.e.

$$(9.6) \quad \mathfrak{h}'_2 + \mathfrak{so}(q_2 - p_1) = \mathfrak{h}_2 = \mathfrak{so}(p_2, q_2).$$

Suppose first that \mathfrak{h}_2 is simple. Then this is a factorization of $\mathfrak{so}(p_2, q_2)$ with one factor compact and we deduce from Lemma 2.4 that $\mathfrak{so}(p_2, q_2)$ is compact as well, i.e. $p_2 = 0$ or $q_2 = 0$. If $q_2 = 0$, then $p_1 = 0$, and $\mathfrak{h}'_2 = \mathfrak{h}_2$. Hence we may assume that $p_2 = 0$ and then

$$\mathfrak{h}'_2 + \mathfrak{so}(q_2 - p_1) = \mathfrak{so}(q_2).$$

We deduce from Proposition 2.5 that $p_1 = 1, 2, 3$ and read off the possibilities (5), (6), (7), (8) for \mathfrak{h}'_2 .

In case \mathfrak{h}_2 is not simple possibilities are $\mathfrak{h}_2 = \mathfrak{so}(2, 2)$ and $\mathfrak{h}_2 = \mathfrak{so}(4)$. Only the latter is possible with (9.6), which in that case gives $p_1 = 1$ and $\mathfrak{h}'_2 \simeq \mathfrak{so}(3)$ and leads to (4).

The case where $\mathfrak{h}'_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}'_1 \subsetneq \mathfrak{h}_1$ is analogous, and leads to the same results but with r and s interchanged. This finishes the treatment of $p_2 \leq q_1$ and (9.4).

We now treat the case of $p_2 \geq q_1$ and (9.5), where $\mathfrak{l}_{\mathfrak{h}} < \mathfrak{h}_2$. Let $\mathfrak{h}' := \mathfrak{h}_1 \oplus \mathfrak{h}'_2 \subset \mathfrak{h}$ with $\mathfrak{h}'_2 < \mathfrak{h}_2$ be a spherical subalgebra of \mathfrak{g} . The condition is that H_2/H'_2 is a real spherical space for the action of L_H . In particular we must have (see Corollary 2.10)

$$(9.7) \quad \mathfrak{h}'_2 + \mathfrak{so}(p_2 - q_1, q_2 - p_1) = \mathfrak{h}_2 = \mathfrak{so}(p_2, q_2)$$

and p_2, q_2 both non-zero. Hence \mathfrak{h}_2 is non-compact and we may assume it is simple. According to Proposition 2.5 we derive that $p_1 + q_1$ equals 1, 2, 3. Suppose first that \mathfrak{h}'_2 is of Type I in \mathfrak{h}_2 , see the four cases in Lemma 5.4. Onishchik's list, Table 2, shows that only $\mathfrak{h}'_2 = \mathfrak{u}(\frac{p_2}{2}, \frac{q_2}{2})$ can be compatible with (9.7), and then $p_1 + q_1 = 1$. Hence $\mathfrak{h} = \mathfrak{h}_2$ and $\mathfrak{h}' = \mathfrak{h}'_2$ is real spherical according to Krämer. Further subalgebras of type $\mathfrak{h}'' := \mathfrak{u}(1) \oplus \mathfrak{h}''_2$ with $\mathfrak{h}''_2 < [\mathfrak{h}'_2, \mathfrak{h}'_2]$ a proper maximal spherical subalgebra are excluded. Indeed (9.7) and Proposition 2.5 only allow $\mathfrak{h}''_2 = \mathfrak{sp}(\frac{p_2}{4}, \frac{q_2}{4})$ and we arrive at the tower

$$\mathfrak{g} = \mathfrak{so}(p_2 + 1, q_2) \supset \mathfrak{u}(\frac{p_2}{2}, \frac{q_2}{2}) \supset \mathfrak{h}'' = \mathfrak{sp}(\frac{p_2}{4}, \frac{q_2}{4}) + \mathfrak{u}(1).$$

Now the real spherical pair $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(p_2 + 1, q_2), \mathfrak{u}(\frac{p_2}{2}, \frac{q_2}{2}))$ has structural algebra $\mathfrak{l}_{\mathfrak{h}} = \mathfrak{u}(\frac{|q_2 - p_2|}{2})$ and we deduce that $(\mathfrak{g}, \mathfrak{h}'')$ is not real spherical by Proposition 2.9 and Proposition 2.5. This leads us to decide whether $\mathfrak{h}_3 := [\mathfrak{h}'_2, \mathfrak{h}'_2] = \mathfrak{su}(\frac{p_2}{2}, \frac{q_2}{2})$ is real spherical. According to Krämer, \mathfrak{h}_3 is not absolutely spherical. Without loss of generality we may assume $p_1 = 1$ and $q_1 = 0$ and then $\mathfrak{l}_{\mathfrak{h}} = \mathfrak{so}(p_2, q_2 - 1)$. Observe that \mathfrak{h}_3 is real spherical if and only if $\mathfrak{h} \cap \mathfrak{m}$ surjects onto the center of \mathfrak{h}'_2 . This is the case precisely when $p_2 \neq q_2$ and $p_2 \neq q_2 - 1$ (see Lemma 9.1), and it leads to cases (2)-(3).

Finally assume that \mathfrak{h}'_2 is of type II or III in \mathfrak{h}_2 . The type II subalgebras appear only for $\mathfrak{h}_2 = \mathfrak{so}(4, 4)$ (see Lemma 5.6) and are excluded by the dimension bound (5.1). This leaves us with the examination of the two type III cases from Lemma 5.7. We begin with $\mathfrak{h}'_2 = \mathfrak{spin}(4, 3)$ in $\mathfrak{h}_2 = \mathfrak{so}(4, 4) = \mathfrak{so}(p_2, q_2)$. Recall that $p_1 + q_1 = 1, 2, 3$ and note that p_1 and q_1 have to differ by three in order for \mathfrak{g} not to be quasisplit. Hence we may assume that $\mathfrak{h}_1 = \mathfrak{so}(3)$, i.e. $\mathfrak{g} = \mathfrak{so}(4, 7)$ and $\mathfrak{h}' = \mathfrak{so}(3) \oplus \mathfrak{spin}(4, 3) \subset \mathfrak{so}(3) \oplus \mathfrak{so}(4, 4)$.

We claim that \mathfrak{h}' is real spherical. For that we need to show that the $L_H = \mathrm{SO}(1, 4)$ -space $H_2/H'_2 = \mathrm{SO}(4, 4)/\mathrm{Spin}(4, 3)$ is real spherical. To this end we lift to $\mathrm{Spin}(4, 4)$, apply the exceptional outer automorphism which swaps the simple roots α_1 and α_4 , and go back to $\mathrm{SO}(4, 4)$. Then $\mathrm{Spin}(4, 3)$ and $\mathrm{SO}(1, 4) \subseteq \mathrm{SO}(2, 4)$ are converted to $\mathrm{SO}(4, 3)$ and $\mathrm{Sp}(1, 1) \subseteq \mathrm{SU}(2, 2)$, respectively. The complexification of this situation is the third case of Table 2 with $n = 2$. Using the last column of the table we get $\mathrm{SO}(4, 4)/\mathrm{SO}(4, 3) = \mathrm{Sp}(1, 1)/\mathrm{Sp}(0, 1)$, which is real spherical as a $\mathrm{Sp}(1, 1)$ -variety by Lemma 9.1. This proves the claim and furnishes case (10).

Next we move on to the case where $\mathfrak{h}'_2 = \mathbf{G}_2^1$ and $\mathfrak{h}_2 = \mathfrak{so}(3, 4)$. As before $p_1 + q_1 = 1, 2, 3$. The case $p_1 + q_1 = 3$ is excluded by the dimension bound and the case with $p_1 + q_1 = 1$ leads to absolutely spherical pairs. The case $p_1 = q_1 = 1$ is quasi-split. This leaves us with $\mathfrak{h}' = \mathfrak{so}(2) \oplus \mathbf{G}_2^1$ in $\mathfrak{g} = \mathfrak{so}(3, 6)$.

We claim that this case is real spherical. Here we have to show that $H_2/H'_2 = \mathrm{SO}(3, 4)/\mathbf{G}_2^1$ is spherical as $L_H = \mathrm{SO}(1, 4)$ -variety. But that follows immediately from the isomorphism $\mathrm{SO}(3, 4)/\mathbf{G}_2^1 \cong \mathrm{SO}(4, 4)/\mathrm{Spin}(3, 4)$ (eighth case of Table 2) and the proof of case (10) above. This yields case (9). \square

Proposition 9.6. *Let $\mathfrak{g} = \mathfrak{sp}(p, q)$. Then every reductive subalgebra is real spherical if and only if it is absolutely spherical or conjugate to one of the following:*

- (1) $\mathfrak{sp}(p-1, q)$,
- (2) $\mathfrak{sp}(p, q-1)$,
- (3) $\mathfrak{su}(p, q)$ with $p \neq q$.

Proof. We need to consider subalgebras of the following cases from (11) and (12) in Table 4:

$$\mathfrak{h} = \mathfrak{sp}(p, \mathbb{C}), \mathfrak{sp}(p_1, q_1) \times \mathfrak{sp}(p_2, q_2), \mathfrak{u}(p, q), \mathfrak{gl}(p, \mathbb{H})$$

where $q = p$ in the first and last cases. Since symplectic algebras admit no factorizations by Proposition 2.5 the first case is excluded with Proposition 2.9. We can argue similarly in the second case, except when $\mathfrak{l}_{\mathfrak{h}}$ surjects onto one of the factors of $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$. According to Table 4 this happens if and only if \mathfrak{h}_1 or \mathfrak{h}_2 is $\mathfrak{sp}(1)$, in which case the other factor \mathfrak{h}' of \mathfrak{h} is $\mathfrak{sp}(p-1, q)$ or $\mathfrak{sp}(p, q-1)$. Then \mathfrak{m} belongs to \mathfrak{h} and surjects onto $\mathfrak{sp}(1)$ along \mathfrak{h}' . Hence \mathfrak{h}' is spherical. Further we observe that it is not absolutely spherical, but any strictly larger subalgebra is. This gives (1)-(2).

For the third case, $\mathfrak{h} = \mathfrak{u}(p, q)$, we note that $\mathfrak{l} \cap \mathfrak{h}$ is compact according to Table 4. Hence if $\mathfrak{h}' \subset \mathfrak{h}$ satisfies (9.1) then Lemma 2.4 implies $\mathfrak{su}(p, q) \subset \mathfrak{h}'$. With Lemma 9.1 we conclude (3).

Finally, in the fourth case $\mathfrak{h} = \mathfrak{gl}(p, \mathbb{H})$ we have $\mathfrak{l}_{\mathfrak{h}} = \mathfrak{u}(1)^p$ and hence no proper factorization is possible. \square

9.2. The exceptional cases. For convenience we record here Cartan's list of the nine symmetric subgroups in the complex exceptional Lie groups of type E:

$G_{\mathbb{C}}$	$E_6^{\mathbb{C}}$	$E_7^{\mathbb{C}}$	$E_8^{\mathbb{C}}$
$H_{1\mathbb{C}}$	$\mathrm{Sp}(4, \mathbb{C})$	$\mathrm{SL}(8, \mathbb{C})$	$\mathrm{SO}(16, \mathbb{C})$
$H_{2\mathbb{C}}$	$\mathrm{SL}(6, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$	$\mathrm{SO}(12, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$	$E_7^{\mathbb{C}} \times \mathrm{SL}(2, \mathbb{C})$
$H_{3\mathbb{C}}$	$\mathrm{SO}(10, \mathbb{C}) \times \mathrm{SO}(2, \mathbb{C})$	$E_6^{\mathbb{C}} \times \mathrm{SO}(2, \mathbb{C})$	
$H_{4\mathbb{C}}$	$F_4^{\mathbb{C}}$		

For the list of real symmetric subgroups we shall refer to [3].

Proposition 9.7. *Let \mathfrak{g} be a non-complex and non-compact simple exceptional Lie algebra and let $\mathfrak{h} < \mathfrak{g}$ be reductive. Then $(\mathfrak{g}, \mathfrak{h})$ is real spherical if and only if it is absolutely spherical or, up to conjugation,*

- (1) $\mathfrak{g} = F_4^2$ and $\mathfrak{h} = \mathfrak{sp}(2, 1) \oplus \mathfrak{f}$ with $\mathfrak{f} \subset \mathfrak{u}(1)$, or
- (2) $\mathfrak{g} = E_6^4$ and $\mathfrak{h} = \mathfrak{sl}(3, \mathbb{H}) \oplus \mathfrak{f}$ with $\mathfrak{f} \subset \mathfrak{u}(1)$, or
- (3) $\mathfrak{g} = E_7^2$ and $\mathfrak{h} = E_6^2$ or E_6^3 .

Proof. If \mathfrak{g} is quasi-split we apply Lemma 2.13. In particular we can then assume $\mathfrak{g}_{\mathbb{C}} \neq G_2^{\mathbb{C}}$.

This leaves us with the E and F-cases which are not quasi-split. As before we shall use the notation \mathfrak{h}' for a given candidate of a real spherical subalgebra of \mathfrak{g} . It follows from Lemma 7.1 that we may assume \mathfrak{h}' is contained in a symmetric subalgebra which we then denote by \mathfrak{h} . We can assume the inclusion is proper.

We use Table 5 for $\mathfrak{l} \cap \mathfrak{h}$, where $Q = LU$ is the adapted parabolic for $Z = G/H$. We note that $\mathfrak{l} \cap \mathfrak{h}$ only depends on the real form \mathfrak{g} and the complexification $\mathfrak{h}_{\mathbb{C}}$ (see Lemma 8.1).

We start with $\mathfrak{g} = F_4^2$. From Table 5 we deduce that either $\mathfrak{h} = \mathfrak{so}(8, 1)$ and $\mathfrak{l}_{\mathfrak{h}} = \mathfrak{so}(7)$, or $\mathfrak{h} = \mathfrak{sp}(2, 1) \oplus \mathfrak{su}(2)$ and $\mathfrak{l}_{\mathfrak{h}} = \mathfrak{so}(4) \oplus \mathfrak{so}(3)$. Since $\mathfrak{so}(9, \mathbb{C})$ does not admit non-trivial factorizations, no proper reductive subalgebra of \mathfrak{h} can be spherical in the first case. In the second case we observe with Remark 8.2 that $\mathfrak{l}_{\mathfrak{h}}$ projects onto $\mathfrak{su}(2)$, the second factor of \mathfrak{h} , and the cases in (1) emerge.

We continue with $\mathfrak{g} = E_8^2$. Any symmetric subalgebra of \mathfrak{g} is a real form of either $\mathfrak{h}_{1,\mathbb{C}}$ or $\mathfrak{h}_{2,\mathbb{C}}$ from the table above. However, no proper reductive subalgebra of \mathfrak{h}_1 can satisfy the dimension bound (7.5). We assume $\mathfrak{h}' \subsetneq \mathfrak{h}_2$, a real form of $\mathfrak{h}_{2,\mathbb{C}}$, and let $Q = LU$ be the adapted parabolic for G/H_2 . Then

$$(9.8) \quad \mathfrak{l} \cap \mathfrak{h}_2 + \mathfrak{h}' = \mathfrak{h}_2$$

by (9.1). Here $(\mathfrak{l} \cap \mathfrak{h}_2)_{\mathbb{C}} = \mathfrak{so}(8, \mathbb{C})$ and the projection of this algebra to the second component of $\mathfrak{h}_{2,\mathbb{C}}$ must be zero as $\mathfrak{so}(8, \mathbb{C})$ is irreducible and of higher dimension than $\mathfrak{sl}(2, \mathbb{C})$. Hence $(\mathfrak{l} \cap \mathfrak{h}_2)_{\mathbb{C}} \subset E_7^{\mathbb{C}}$. However by Proposition 2.5, $E_7^{\mathbb{C}}$ admits no proper factorizations, and hence we must have $E_7^{\mathbb{C}} \subset \mathfrak{h}'$. Thus $\mathfrak{l} \cap \mathfrak{h}_2 \subset \mathfrak{h}'$ which contradicts (9.8).

Next we investigate the real forms of $\mathfrak{g}_{\mathbb{C}} = E_7^{\mathbb{C}}$. According to the table above the symmetric subalgebras in $\mathfrak{g}_{\mathbb{C}}$ are $\mathfrak{h}_{i,\mathbb{C}}$, $i = 1, 2, 3$.

We start with $\mathfrak{g} = E_7^2$ and note that the dimension bound (7.3) excludes that $\mathfrak{h} \subsetneq \mathfrak{h}_1$. Next we consider the pair $(\mathfrak{g}, \mathfrak{h}_2)$ and the corresponding adapted parabolic $Q = LU$. Assume $\mathfrak{h} \subsetneq \mathfrak{h}_2$, then (9.8) holds as before. Here $(\mathfrak{l} \cap \mathfrak{h}_2)_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})^3$ embeds in the first component $\mathfrak{so}(12, \mathbb{C})$ of $\mathfrak{h}_{2,\mathbb{C}}$ (see Remark 8.2). By Proposition 2.5 there exists no proper factorization of $\mathfrak{so}(12, \mathbb{C})$ with $\mathfrak{sl}(2, \mathbb{C})^3$ as a factor, and hence we conclude that $\mathfrak{so}(12, \mathbb{C}) \subset \mathfrak{h}_{\mathbb{C}}$. Thus $\mathfrak{l} \cap \mathfrak{h}_2 \subset \mathfrak{h}$ which contradicts (9.8).

For the third case we note that $E_6^{\mathbb{C}}$ has codimension 1 in $\mathfrak{h}_{3,\mathbb{C}}$ and does not admit proper factorizations by Proposition 2.5. Hence if $\mathfrak{h} \subsetneq \mathfrak{h}_3$ then $\mathfrak{h}_{\mathbb{C}} = E_6^{\mathbb{C}}$. We deduce that \mathfrak{h} is real spherical from Lemma 9.1. This gives (3).

We move on with $\mathfrak{g} = E_7^3$, and start with the assumption that $\mathfrak{h} \subsetneq \mathfrak{h}_1$. Here $Q = P$ by Lemma 2.11 and this implies that $\dim H_1/H_1 \cap L = \dim G/P = 51$, and hence $\dim H_1 \cap L = 12$. According to Proposition 2.5 the only factorizations for \mathfrak{h}_1 are given by

- $(\mathfrak{sl}(8, \mathbb{C}), \mathfrak{sp}(4, \mathbb{C}), \mathfrak{sl}(7, \mathbb{C}))$,
- $(\mathfrak{sl}(8, \mathbb{C}), \mathfrak{sp}(4, \mathbb{C}), \mathfrak{s}(\mathfrak{gl}(1, \mathbb{C}) \oplus \mathfrak{gl}(7, \mathbb{C})))$,

and none of these factors have dimension 12. With Proposition 2.9 we reach a contradiction.

For the case of \mathfrak{h}_2 we first recall $\mathfrak{h}_2 \cap \mathfrak{l} = \mathfrak{so}(6) \oplus \mathfrak{so}(2) \oplus \mathfrak{sl}(2, \mathbb{R})$. It follows from Remark 8.2 that $\mathfrak{h}_2 \cap \mathfrak{l}$ does not surject onto the $\mathfrak{sl}(2)$ -component of \mathfrak{h}_2 . Hence no proper reductive subalgebras of \mathfrak{h}_2 can be real spherical.

For \mathfrak{h}_3 we are again left to check whether a real form of the first component $E_6^{\mathbb{C}}$ of $\mathfrak{h}_{3,\mathbb{C}}$ is real spherical. Here $(\mathfrak{l} \cap \mathfrak{h}_2)_{\mathbb{C}} = \mathfrak{so}(8, \mathbb{C})$ and thus the projection of $(\mathfrak{l} \cap \mathfrak{h}_2)_{\mathbb{C}}$ to the second component of \mathfrak{h}_3 is trivial, and hence no real form of $E_6^{\mathbb{C}}$ can be spherical.

Finally we consider $\mathfrak{g}_{\mathbb{C}} = E_6^{\mathbb{C}}$ and $\mathfrak{g} = E_6^3$ or E_6^4 . The complex symmetric subalgebras $\mathfrak{h}_{1,\mathbb{C}}$, $i = 1, \dots, 4$, are given in the table.

Since both \mathfrak{h}_1 and \mathfrak{h}_4 admit no factorizations there are no reductive real spherical subalgebras which are contained in \mathfrak{h}_1 or \mathfrak{h}_4 . We move on and assume $\mathfrak{h} \subsetneq \mathfrak{h}_2$, where \mathfrak{h}_2 is a real form of $\mathfrak{h}_{2,\mathbb{C}}$ in \mathfrak{g} . Write $\mathfrak{h}_2 = \mathfrak{h}'_2 \oplus \mathfrak{h}''_2$ with $\mathfrak{h}'_{2,\mathbb{C}} = \mathfrak{sl}(6, \mathbb{C})$ and $\mathfrak{h}''_{2,\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$. From Table 5 we infer that

$$(9.9) \quad \mathfrak{l} \cap \mathfrak{h}_2 = \mathfrak{u}(2) \oplus \mathfrak{u}(2) \quad \text{for} \quad \mathfrak{g} = E_6^3$$

$$(9.10) \quad \mathfrak{l} \cap \mathfrak{h}_2 = \mathfrak{so}(5) \oplus \mathfrak{so}(3) \oplus \mathfrak{gl}(1, \mathbb{R}) \quad \text{for} \quad \mathfrak{g} = E_6^4$$

We claim that \mathfrak{h} is not spherical for $\mathfrak{g} = E_6^3$. Otherwise, according to Propositions 2.5, 2.9, \mathfrak{h}_2 would surject to a factor of \mathfrak{h}''_2 . But this is not possible by Remark 8.2.

For $\mathfrak{g} = E_6^4$, we claim that $\mathfrak{l} \cap \mathfrak{h}_2$ surjects onto $\mathfrak{h}_2'' = \mathfrak{su}(2)$ and in particular that $\mathfrak{h}_2' = \mathfrak{sl}(3, \mathbb{H})$ is real spherical. In order to establish that we let $V \subset \mathfrak{g}_{\mathbb{C}}$ be the orthogonal complement of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Note that $\dim_{\mathbb{C}} V = 40$ and that V is an irreducible module for $\mathfrak{h}_{\mathbb{C}}$. Hence $V = \bigwedge^3 \mathbb{C}^6 \otimes \mathbb{C}^2$ as an $\mathfrak{h}_{2, \mathbb{C}}$ -module. Notice that $\mathfrak{a} := V \cap \mathfrak{z}(\mathfrak{l}) \neq \{0\}$ and that \mathfrak{a} is fixed under $\mathfrak{l}_{\mathfrak{h}}$. In order to obtain a contradiction, assume that $\mathfrak{l}_{\mathfrak{h}_2} \subset \mathfrak{h}_2'$. Then, as an $\mathfrak{l}_{\mathfrak{h}}$ -module, $V = \bigwedge^3 \mathbb{C}^6 \oplus \bigwedge^3 \mathbb{C}^6$. Since $V^{\mathfrak{l}_{\mathfrak{h}}} \neq \{0\}$ we deduce that the irreducible $\mathfrak{h}_{2, \mathbb{C}}' = \mathfrak{sl}(6, \mathbb{C})$ -module $\bigwedge^3 \mathbb{C}^6$ is spherical for $[\mathfrak{l} \cap \mathfrak{h}_2, \mathfrak{l} \cap \mathfrak{h}_2]_{\mathbb{C}} \simeq \mathfrak{sp}(2, \mathbb{C}) \oplus \mathfrak{sp}(1, \mathbb{C})$. But $\bigwedge^3 \mathbb{C}^6$ decomposes under $\mathfrak{sp}(3, \mathbb{C})$ into $V(\omega_1) \oplus V(\omega_3)$ and hence is not spherical for the pair $(\mathfrak{sp}(3, \mathbb{C}), \mathfrak{sp}(2, \mathbb{C}) \oplus \mathfrak{sp}(1, \mathbb{C}))$ by [16], Tabelle 1. This gives the desired contradiction, and hence (2).

Finally we come to the case of $\mathfrak{h}_{3, \mathbb{C}}$. Here it is known that $\mathfrak{h}_{3, \mathbb{C}}' = \mathfrak{so}(10, \mathbb{C})$ is a complex spherical subgroup by [16]. From the list in Proposition 2.5 we extract that the only factorizations of $\mathfrak{so}(10, \mathbb{C})$ are given by $(\mathfrak{so}(10, \mathbb{C}), \mathfrak{so}(9, \mathbb{C}), \mathfrak{sl}(5, \mathbb{C}) + \mathfrak{f})$, $\mathfrak{f} \subset \mathfrak{u}(1)$. Now for $\mathfrak{g} = E_6^3, E_6^4$ we have $[\mathfrak{l} \cap \mathfrak{h}_3, \mathfrak{l} \cap \mathfrak{h}_3]_{\mathbb{C}}$ is $\mathfrak{so}(6, \mathbb{C})$ or $\mathfrak{spin}(7, \mathbb{C})$ and the factorization of $\mathfrak{h}_{3, \mathbb{C}}$ is not possible. This concludes the proof of the proposition. \square

9.3. The complex cases.

Proposition 9.8. *Let \mathfrak{g} be a complex simple Lie algebra and $\mathfrak{h} < \mathfrak{g}$ a real reductive subalgebra. Then $(\mathfrak{g}, \mathfrak{h})$ is real spherical if and only if one of the following holds*

- (1) \mathfrak{h} is a real form of \mathfrak{g} ,
- (2) \mathfrak{h} is a complex spherical subalgebra of \mathfrak{g} ,

or \mathfrak{h} is conjugate to $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ with

- (3) $(\mathfrak{h}_1, \mathfrak{h}_2) = (\mathfrak{z}(\mathfrak{h}), [\mathfrak{h}, \mathfrak{h}])$, $\dim_{\mathbb{R}} \mathfrak{h}_1 = 1$, and $(\mathfrak{g}, \mathfrak{h}_2)$ is one of the following complex spherical pairs
 - (a) $(\mathfrak{sl}(n+m, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C}) \times \mathfrak{sl}(m, \mathbb{C}))$, $0 < m < n$,
 - (b) $(\mathfrak{sl}(2n+1, \mathbb{C}), \mathfrak{sp}(n, \mathbb{C}))$,
 - (c) $(\mathfrak{so}(2n, \mathbb{C}), \mathfrak{sl}(n, \mathbb{C}))$, n odd,
 - (d) $(E_6^{\mathbb{C}}, \mathfrak{so}(10, \mathbb{C}))$,
- (4) $\mathfrak{h}_1 = \mathfrak{su}(2)$ or $\mathfrak{sl}(2, \mathbb{R})$, and $(\mathfrak{g}, \mathfrak{h}_2) = (\mathfrak{sp}(n+1, \mathbb{C}), \mathfrak{sp}(n, \mathbb{C}))$.

Proof. First observe that \mathfrak{g} , considered as a real Lie algebra, is quasisplit. Hence $\mathfrak{h} < \mathfrak{g}$ is real spherical if and only if $\mathfrak{h}_{\mathbb{C}} < \mathfrak{g}_{\mathbb{C}}$ is spherical. We may identify $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g} \oplus \mathfrak{g}$. Now according to [18], the complex spherical subalgebras $\tilde{\mathfrak{h}}$ of $\mathfrak{g} \oplus \mathfrak{g}$ are given as follows

- (1) $\tilde{\mathfrak{h}} = \text{diag}(\mathfrak{g})$.
- (2) $\tilde{\mathfrak{h}} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}_i < \mathfrak{g}$ complex spherical.
- (3) There exists a complex spherical subalgebra $\mathfrak{h}_0 < \mathfrak{g}$ with $\mathfrak{z}(\mathfrak{h}_0) \neq 0$, $[\mathfrak{h}_0, \mathfrak{h}_0]$ complex spherical and $\tilde{\mathfrak{h}} = [\mathfrak{h}_0, \mathfrak{h}_0] \oplus \mathfrak{z}(\mathfrak{h}_0) \oplus [\mathfrak{h}_0, \mathfrak{h}_0]$ and $\mathfrak{z}(\mathfrak{h}_0)$ diagonally embedded.
- (4) $\mathfrak{g} = \mathfrak{sp}(n+1, \mathbb{C})$ and $\tilde{\mathfrak{h}} = \mathfrak{sp}(n, \mathbb{C}) \oplus \mathfrak{sp}(1, \mathbb{C}) \oplus \mathfrak{sp}(n, \mathbb{C})$ with $\mathfrak{sp}(1, \mathbb{C})$ diagonally embedded.

It is easy to see that these four cases restricted to $\tilde{\mathfrak{h}} = \mathfrak{h}_{\mathbb{C}}$ correspond to the four cases listed in the proposition. \square

By combining Propositions 9.2, 9.3, 9.4, 9.5, 9.6, 9.7, 9.8, and Table 7 we finally obtain Theorem 1.1.

10. APPENDIX: NON-SYMMETRIC REAL FORMS OF KRÄMER'S LIST

In Table 6 below is the list, according to Krämer [16], of the non-symmetric complex spherical subalgebras $\mathfrak{h}_{\mathbb{C}}$ of a complex simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ up to isomorphism. All algebras are implied to be complex.

	$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{h}_{\mathbb{C}}$		$\bar{\mathfrak{h}}_{\mathbb{C}}$
(1)	$\mathfrak{sl}(m+n)$	$\mathfrak{sl}(m) + \mathfrak{sl}(n)$	$m > n \geq 1$	$\mathfrak{sl}(\mathfrak{gl}(m) + \mathfrak{gl}(n))$
(2)	$\mathfrak{sl}(2n+1)$	$\mathfrak{sp}(n) + \mathfrak{f}$	$n \geq 2, \mathfrak{f} \subseteq \mathbb{C}$	$\mathfrak{gl}(2n)$
(3)	$\mathfrak{sp}(n)$	$\mathfrak{sp}(n-1) + \mathbb{C}$	$n \geq 3$	$\mathfrak{sp}(n-1) + \mathfrak{sp}(1)$
(4)	$\mathfrak{so}(2n)$	$\mathfrak{sl}(n)$	$n \geq 5$ odd	$\mathfrak{gl}(n)$
(5)	$\mathfrak{so}(2n+1)$	$\mathfrak{gl}(n)$	$n \geq 2$	$\mathfrak{so}(2n)$
(6)	$\mathfrak{so}(9)$	$\mathfrak{spin}(7)$		$\mathfrak{so}(8)$
(7)	$\mathfrak{so}(10)$	$\mathfrak{spin}(7) + \mathbb{C}$		$\mathfrak{so}(8) + \mathbb{C}$
(8)	G_2	$\mathfrak{sl}(3)$		—
(9)	$\mathfrak{so}(7)$	G_2		—
(10)	$\mathfrak{so}(8)$	G_2		$\mathfrak{so}(7)$
(11)	E_6	$\mathfrak{spin}(10)$		$\mathfrak{spin}(10) + \mathbb{C}$

Table 6

Inspecting this table one realizes that it has a certain structure. In all cases but (8) and (9) (which will be handled by ad hoc arguments) there is a canonical intermediate subalgebra $\bar{\mathfrak{h}}_{\mathbb{C}}$, given in the last column, with the following properties (a)-(c).

- (a) The pair $(\mathfrak{g}_{\mathbb{C}}, \bar{\mathfrak{h}}_{\mathbb{C}})$ is symmetric. Hence all of its real forms appear up to isomorphism in Berger's list.
- (b) Except for case (10), the pair $(\bar{\mathfrak{h}}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is symmetric, as well. Even though $\bar{\mathfrak{h}}_{\mathbb{C}}$ may not be simple the real forms of these pairs are easily read off from Berger's list.
- (c) If $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is defined over \mathbb{R} then also $\bar{\mathfrak{h}}_{\mathbb{C}}$ is defined over \mathbb{R} . Indeed, $\bar{\mathfrak{h}}_{\mathbb{C}} = N_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{h}_{\mathbb{C}})$ in cases (1), (4), and (11). Moreover, $\bar{\mathfrak{h}}_{\mathbb{C}} = N_{\mathfrak{g}_{\mathbb{C}}}(C_{\mathfrak{g}_{\mathbb{C}}}([\mathfrak{h}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}]))$ in cases (2), (3), and (7). For cases (5) and (6) one argues as follows: for all real forms of $\mathfrak{g}_{\mathbb{C}}$, the defining representation V is defined over \mathbb{R} . Then $\bar{\mathfrak{h}}_{\mathbb{C}} = C_{\mathfrak{g}_{\mathbb{C}}}(V^{\mathfrak{h}_{\mathbb{C}}})$ is defined over \mathbb{R} , as well. In case (10) the isomorphism $\mathfrak{so}^*(8) \cong \mathfrak{so}(6, 2)$ (via triality) shows that it suffices to consider real forms for which V is defined over \mathbb{R} . Then the argument above works.

Using the reasoning above, we obtain almost all items in the following Table 7 of non-symmetric, absolutely spherical reductive subalgebras \mathfrak{h} of absolutely simple non-compact Lie algebras \mathfrak{g} up to isomorphism. Only the last five rows, which relate to (8), (9) and (10) above, require a separate argument.

The cases involving real forms of G_2 are handled using the following remarks. The maximal compact subalgebra of G_2^1 is $\mathfrak{su}(2) + \mathfrak{su}(2)$. Hence $\mathfrak{su}(3) \not\subseteq G_2^1$. Moreover, the invariant scalar product on the 7-dimensional representation of G_2 and G_2^1 has signature (7, 0) and (4, 3), respectively. In the second one the isotropy group of a vector with positive or negative square length is $SL(3, \mathbb{R})$ or $SU(2, 1)$, respectively. This gives the pairs related to (8) and (9), and finally (10) can be reduced to case (9) in the same way as (b) above.

\mathfrak{g}	\mathfrak{h}	
$\mathfrak{sl}(m+n, \mathbb{R})$	$\mathfrak{sl}(m, \mathbb{R}) + \mathfrak{sl}(n, \mathbb{R})$	$m > n \geq 1$
$\mathfrak{su}(p_1 + p_2, q_1 + q_2)$	$\mathfrak{su}(p_1, q_1) + \mathfrak{su}(p_2, q_2)$	$p_1 + q_1 > p_2 + q_2 \geq 1$
$\mathfrak{sl}(m+n, \mathbb{H})$	$\mathfrak{sl}(m, \mathbb{H}) + \mathfrak{sl}(n, \mathbb{H})$	$m > n \geq 1$
$\mathfrak{sl}(2n+1, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R}) + \mathfrak{f}$	$n \geq 2, \mathfrak{f} \subseteq \mathbb{R}$
$\mathfrak{su}(p, q)$	$\begin{cases} \mathfrak{sp}(\frac{p-1}{2}, \frac{q}{2}) + \mathfrak{f} \\ \mathfrak{sp}(p-1 = q, \mathbb{R}) + \mathfrak{f} \end{cases}$	$p \text{ odd}, q \text{ even}, \mathfrak{f} \subseteq i\mathbb{R}$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sp}(n-1, \mathbb{R}) + \mathfrak{f}$	$n \geq 3, \mathfrak{f} \in \{\mathbb{R}, i\mathbb{R}\}$
$\mathfrak{sp}(p, q)$	$\mathfrak{sp}(p-1, q) + i\mathbb{R}$	$p+q \geq 3$
$\mathfrak{so}(2p, 2q)$	$\mathfrak{su}(p, q)$	$p+q \text{ odd}$
$\mathfrak{so}^*(2n)$	$\mathfrak{su}(p, q)$	$n = p+q \text{ odd}$
$\mathfrak{so}(p, q)$	$\begin{cases} \mathfrak{su}(\frac{p-1}{2}, \frac{q}{2}) + i\mathbb{R} \\ \mathfrak{sl}(\frac{p-1}{2} = \frac{q}{2}, \mathbb{R}) + \mathbb{R} \end{cases}$	$p \text{ odd}, q \text{ even}$
$\mathfrak{so}(5, 4)$	$\mathfrak{spin}(4, 3)$	
$\mathfrak{so}(8, 1)$	$\mathfrak{spin}(7, 0)$	
$\mathfrak{so}(5, 5)$	$\mathfrak{spin}(4, 3) + \mathbb{R}$	
$\mathfrak{so}(6, 4)$	$\mathfrak{spin}(4, 3) + i\mathbb{R}$	
$\mathfrak{so}(8, 2)$	$\mathfrak{spin}(7, 0) + i\mathbb{R}$	
$\mathfrak{so}(9, 1)$	$\mathfrak{spin}(7, 0) + \mathbb{R}$	
$\mathfrak{so}^*(10)$	$\mathfrak{spin}(6, 1) + \mathbb{R}, \mathfrak{spin}(5, 2) + \mathbb{R}$	
E_6^1	$\mathfrak{so}(5, 5)$	
E_6^2	$\mathfrak{so}(6, 4), \mathfrak{so}^*(10)$	
E_6^3	$\mathfrak{so}(10), \mathfrak{so}(8, 2), \mathfrak{so}^*(10)$	
E_6^4	$\mathfrak{so}(9, 1)$	
G_2^1	$\mathfrak{sl}(3, \mathbb{R}), \mathfrak{su}(2, 1)$	
$\mathfrak{so}(4, 3)$	G_2^1	
$\mathfrak{so}(4, 4)$	G_2^1	
$\mathfrak{so}(5, 3)$	G_2^1	
$\mathfrak{so}(7, 1)$	G_2	

Table 7

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